Discount Rates as a Function of Log Size and Valuation Error Measurement

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I. Introduction

This article serves three purposes:

(1) It updates the data in my original article.[1]

(2) It contains further thoughts and insights that have come to me over the past two years.

(3) It is the first article of which I am aware that addresses measurement of valuation error

This article is difficult to understand without having read the original article. For the sake of brevity, I assume that you have read the original article, and I will repeat little from it.

II. Table I: Regression of NYSE Data

Table I contains the same New York Stock Exchange (NYSE) data as Table I in the original article, updated for the past two years, i.e., 1994-1995.

Data Columns

Both Regression #1 and #2 have three columns of data. The first two columns contain regression results for the years 1926-1995 and 1926-1993, respectively, using all 10 deciles[2] in the regression results. The second column allows us to compare the results as reported in my original article with the updated results.

[1] "A Breakthrough in Calculating Reliable Discount Rates," *Valuation*, August, 1994

[2] These are the CRSP decile breakdowns that appear in Ibbotson's SBBI-1996 Yearbook. Each decile is 10% of the NYSE as ordered by market capitalization (size).

The third column of data is for 1926-1995 only, and it excludes Decile #10 data from the regression. The purpose of this column is to evaluate decile #10 as a potential outlier to be excluded.

Regression Equation #1

In this regression, the arithmetic average returns by market capitalization decile is the dependent variable and the standard deviation of returns is the independent variable.

The regression constant was 4.98% for 1926-1993 and rose to 5.24% for 1926-1995. The regression constant is the return that would occur for a zero-risk stock, a proxy for which is a long-term Treasury bond. Indeed, the long-term bond returns very closely match the regression results. The 68 and 70 year average bond returns are 5.02% and 5.17%, respectively, validating the regression constant. Figure 1 graphs this relationship.

There are two problems with using the first regression:

(1) We would need a standard deviation of returns in order to forecast a discount rate. That excludes privately-traded firms, as a firm must have an objective stock price to have a standard deviation of returns.

(2) Even for a publicly-traded firm, which has a standard deviation of returns, we would need to develop a mathematical relationship of firm returns and standard deviations to the decile returns and standard deviations. Note that Columns 2 and 3 in Table I are for diversified portfolios comprised of 169 or 170 firms. The only manner in which the portfolios are not fully diversified is with respect to size.

Regression #2: The "Abrams Model"

The second regression is the more useful one, certainly for valuing privately-held firms. As one would expect, there is not much change in two years. The regression constant decreased from 49.27% to 47.94%, while the slope, i.e., the x-coefficient, increased from -.0163 to -.0157. Figure 2 is a graph of this relationship.

R^2 has declined from 94.2% to 91.4%. The reason for this is that 1994 and 1995 were volatile years, with 0% and 36.1% returns, respectively, on the valued-weighted NYSE.[3] In 1995, the very high returns were disproportionately represented in the two largest deciles. Returns in deciles #1 and #2 were 39.2% and 34.8%, respectively, while returns in deciles #9 and #10

[3] SBBI-1996, page 133

were 18.9 and 25.1%, respectively. These results are contrary to the long run trend of the smallest firms receiving the highest returns.

The standard error of the y estimate increased from 0.77% to 0.89%. While this change may appear minor, its effect will be important later on in this article in evaluating confidence intervals.

In our discussion of Table VI, Column 4 later on in this article, we will discuss the significance of the third column.

Regression #3

The third regression shows the relationship between the decile returns and the decile betas. This is not a proper regression, *per se*, but it does provide some very interesting insights.

The y-intercept should be the risk-free rate and the x-coefficient should be the long-run equity premium of 7.4%[4] if the Capital Asset Pricing Model (CAPM) equation is correct. Instead, the y-intercept is -4.6%, while it should have been 5.2%, and the x-coefficient is 17.14%, while it should have been 7.4%. This demonstrates that CAPM is off by a country mile. Later on you will see that in another form.

While this equation is contrary to the theoretical CAPM, it does constitute an Empirical CAPM, which we could use for a firm whose capitalization is at least as large as a decile #10 firm. Merely select the appropriate decile, use the beta of that decile, possibly with adjustments, and use Regression Equation #3 to generate a discount rate.

Table I, Columns 9-15

These columns appear as they did in the original article on the second page of Table I. In these columns we calculate the standard errors for CAPM and the Abrams Model as 2.72% and 0.89%, respectively. Note that the CAPM standard error is more than three times larger than the log-size standard error and that the latter matches the spreadsheet's own calculation of the standard error on the first page.

III. Table II

Table II shows the implied discount rate for firms of various sizes using Regression Equation #2. A $10 billion firm had a 70-year average return of 11.7%, while a $50 million firm had average

[4] SBBI-1996, page 161

returns of 20.0%. While those values and all values in between are interpolations based on Regression Equation #2, the discount rates for firm values below that are extrapolations.

Figure 4

The implied discount rate column incorporates the updated regression equation. Figure 4 is to the right of Table II, and it graphs returns as a function of the absolute fair market value (FMV), i.e., this is not on a log scale. Since the equation is r = .4794 - .015733 ln FMV, we begin at the extreme left with a return of 48% for a firm worth $1 and subtract the fraction of the ln FMV. A log curve has the shape as in Figure 3, and subtracting a curve of that shape from a constant results in a curve of the shape of Figure 4.

An Explanation of Logarithms & Figure 3

Briefly reviewing some concepts from calculus, the natural logarithm (ln) of the variable x is the area under the curve y = 1/t, where t varies from 1 to x. Its formula is:

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

In our case, x is FMV, so ln FMV is the area under the curve that is the reciprocal of fair market value, y = 1/FMV. The ln of 1 = 0, and ln e = 1 (e is the natural exponent, a transcendental number = 2.718…).

It takes very large changes in FMV to produce a small change in ln FMV. For example, look at Table I, Columns 7 and 8. Ln $48.4 million (Decile 10) = 17.6941, while ln $18.4 billion (Decile 1) is only 23.6358. That is true because the area under the curve 1/FMV, where FMV varies from $48.4 million to $18.4 billion, is only about $6. It is this property of logarithms that we will later see is necessary to produce the mathematical convergence that we need in the iterative process to arrive at a value that is consistent with the underlying assumptions.

An important property of logarithms is that ln xy = ln x + ln y. Since regression equation #2 has the form r = a + b ln FMV, where a = .4794 and b = -.0157334352, we can ask how the discount rate varies with different orders of magnitude of value. For the moment, we will work through some general equations where we vary the value of the firm K times.

Let:

r_1 = the discount rate for Firm #1, whose value = FMV_1
r_2 = the discount rate for Firm #2, whose value = $FMV_2 = K\ FMV_1$

[1]	r_1	$= a + b \ln FMV_1$	Regression Equation #2 applied to Firm #1
[2]	r_2	$= a + b \ln (K\, FMV_1)$	Regression Equation #2 applied to Firm #2
[3]	r_2	$= a + b [\ln K + \ln FMV_1]$	
[4]	r_2	$= a + b \ln FMV_1 + b \ln K$	Here we rearranged [3]
[5]	r_2	$= r_1 + b \ln K$	

In words, the discount rate of a firm K times larger (smaller) than Firm #1 is always b ln K smaller (larger) than r_1.

Let's do some specific examples. First let's see what happens with orders of magnitude of 10. Ln 10 = 2.302535, so b ln 10 = -.0157334352 * 2.302535 = -.0362, or -3.62%. This means that if Firm #2 is 10 times larger (smaller) than Firm #1, its discount rate should be 3.62% lower (higher) than the Firm #1 discount rate. You can see this in Table II. The $10 billion firm has a discount rate of 11.7%, while the $1 billion firm has a discount rate of 15.3%, which is 3.6% higher. The $100 million firm has a discount rate of 19.0%, which is 3.7% higher than the $1 billion firm.[5] Because of the mathematical properties of logarithms, the same *percentage* change in FMV will always result in the same *absolute* change in the discount rate. We also see this phenomenon on graphs with log scales. Equal distances on a log scale are equal percentage changes, not absolute changes.

Let's try one more useful calculation—an order of magnitude 2. Ln 2 = .6931, so b ln K = -.0157334352 * .6931 = -1.09%. Doubling (halving) the value of the firm reduces (increases) the discount rate by 1.09%. You can see that in going from a $10 million firm to a $5 million firm, where the discount rate increased from 22.6% to 23.7%, a 1.1% difference.

Now it is possible to construct your own table. All you need to know is your starting FMV and discount rate. The rest follows easily from the above formulas. Also, we can easily interpolate the table. Suppose you wanted to know the discount rate for a $25 million firm. Simply start with the $50 million firm, where r = 20.0%, and add 1.1% = 21.1%.

Eliminating The Marketable Majority FMV Column

Table II corrects and updates Table II from the original article. Since the original article, I have realized that we should eliminate the "Marketable Majority FMV" column. The reason for this is that stock returns in the market are marketable minority interests. Therefore we must achieve consistency between the *a priori* assumption of the value of the firm, its related discount rate, and the *ex-post* value that results from our valuation, with all numbers being marketable minority interests. Only after we achieve this consistency should we then apply a control premium. We will illustrate this later with examples in Tables IV-A, IV-B, and IV-C.

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[5] There is a slight rounding error

Eliminating The Implied Equity Premium Column

I have also eliminated the implied equity premium column from Table II in the original article to produce Table II in this article. The reason I had this column in the original article is that I was still suffering from the CAPM paradigm, i.e., always thinking in terms of equity premiums.

Table III shows the correlation of bond returns to stock returns is a mere 18.7%, and the correlation of Long Term Treasury yields to stock returns is non-existent at -0.5%. It is actually the latter that is the most relevant.

Unlike most CAPM studies, we are not generating a time series. Instead, we use long term averages. The high R^2 that we get would be overstated in the context of a time series, but that is not relevant in our context. While portfolio managers on Wall Street live or die with next year's stock market returns, our valuations do not. That is true for both the Subject Company's and the NYSE's next year results. Valuation of private businesses is a long-term proposition, and therefore the high R^2 is accurate and appropriate.[6]

Table III[7] shows that when current Long Term Treasury yields are higher or lower than their long run averages, that does not in the least bit imply that stock returns that year will also be correspondingly high or low. Therefore it makes more sense to directly use the regression calculated rates of return and eliminate the equity premium concept rather than artificially split stock returns into the treasury bond returns and equity premiums. Computationally, this is also simpler to use.

IV. Tables IV-A, IV-B, and IV-C: Discounted Cash Flow Valuations

Tables IV-A, IV-B, and IV-C correct and supersede Tables IV-A and IV-B in the original article. These constitute a single valuation using the Regression #2 equation.

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[6] The regression equation is obviously far less accurate at forecasting returns for a *particular* year. With respect to that, the R^2 is overstated. For a comprehensive article on the size effect (and others), see "Disentangling Equity Return Regularities: New Insights and Investment Opportunities," by Bruce I. Jacobs and Kenneth N. Levy, Financial Analysts Journal, May-June 1988. I thank Paul Kaplan of Ibbotson Associates for making me aware of this article and sending it to me. It is very statistical and difficult reading, but it has fascinating insights primarily appropriate for public company valuation. Jacobs and Levy did find the log size effect, among others. However their focus on public companies renders it only slightly useful for valuation of private companies. Their most significant other result for valuation of private firms is they found no industry effect at all—something I have long contended is true. With size explaining so much of returns, there is little left for industry to explain.

[7] Table III in this article did not appear in the original article

Table IV-A

The basic assumptions appear in Rows 1-7. We assume the firm had $100,000 cash flow in 1995. We forecast annual growth in Row 3 and perpetual growth at 6% after the year 2000 (Row 5).

In Row 4, we assume a 30% discount rate. This discount rate is incorrect, but that is fine. Valuation using the Abrams Equation is an iterative process.

The Discounted Cash Flow analysis in Rows 11-23 is standard and requires little explanation other than the Present Value Factors are midyear, and Row 23 is a marketable minority interest value. This is the value from which we must ascertain the consistency of the discount rate assumed and the result that we have achieved.

We now skip to Row 28, where we begin the calculations of the discount rate using the Abrams Equation. In Row 29, we show ln (567,063) = 13.2482. This is the natural log of the marketable minority value of the firm. In Row 30, we multiply that by the x coefficient from the regression, or -.0157334352, to come to -.2084. We add that to the regression constant of .4794, which appears in Row 31, to come to an implied discount rate of 27% (rounded).

This tells us that we initially assumed too high a discount rate, which means that we undervalued the firm. Therefore, Rows 24-27, which contain the control premium and discount for lack of marketability, are irrelevant in this table.

Table IV-B

This table is identical to Table IV-A, except that we now use the 27% discount rate from Table IV-A, Row 32 as our assumed discount rate in Row 4. This gives us a new marketable minority interest value in Row 23 of $642,292. When we do the calculations of the discount rate in Rows 28-32, we find that we have achieved consistency. The new value implies a 27% discount rate, which matches our assumption.

Table IV-C: Adding Specific Company Adjustments To The DCF Analysis

If there are no Specific Company Adjustments, then we would proceed with the calculations in Rows 24-27. For illustrative purposes, however, let's assume that there is only one owner of this firm. He is 62 years old and had a heart attack three years ago. The success of the firm depends to a great deal on his personal relationships with his customers, and that may not be easy to duplicate with a new owner. We decide to add a 2% Specific Company Adjustment to the discount rate.

It is important to achieve internal consistency as we did in Table IV-B before we add Company Specific Adjustments. Now we merely add the 2% to get a 29% discount rate, which we plug into Row 4. The remainder of the table is identical to its predecessors, except that we eliminate the *ex-post* calculation of the discount rate in Rows 28-32, since we are already consistent.

Only now is it appropriate to pay any attention to the control premium and discount for lack of marketability, which appear in Rows 24-26. Our final fair market value is $477,866, as per Row 27.

In a valuation report, it would be unnecessary to show Table IV-A. I would show Tables IV-B and IV-C only.

V. The Nature of the Iterative Process

It is extremely rare to require more than two iterations to achieve consistency in the *ex-ante* and *ex-post* values. The reason is that even if we guess the value of the firm incorrectly by a factor of 10, we will only be 3.6% (more properly, 360 basis points) off in our discount rate. By the time we come to the second iteration, we usually are consistent. The reason behind this is the discount rate being based on the logarithm of the value. As we saw earlier, there is not much difference between the log of $10 billion and the log of $10 million, and multiplying that by the x-coefficient of -.015 further reduces the effects of an initial incorrect estimate of value. This is a convergent system 99% of the time with any kind of reasonable initial guess of value and most unreasonable guesses.

The simplest valuation is that of a firm with constant growth to perpetuity. We can apply the Gordon Model formula to our forecast of cash flow for the coming year. For simplicity, we will use the end-year Gordon Model formula, which is not as accurate as the midyear formula, but for didactic purposes it will serve us better.

We will use the following definitions:

CF = Cash Flow in year t+1 (the first forecast year)
a = .4794, the regression constant from Regression #2
b = -.0157334352, the x-coefficient from Regression #2
V = Fair Market Value (FMV) of the firm
r = the discount rate

The FMV of the firm is:

[6] $V = CF / (r - g)$
[7] $r = a + b \ln V$ Substituting [7] into [6], we get:

$$[8] \qquad V = \quad CF / (a + b \ln V - g)$$

Equation [8] is a transcendental equation with no analytic solution.[8] The simple iterative procedure in Tables IV-A, IV-B, and IV-C are very easy to use and work in almost all situations.

Enter Isaac Newton Into Valuation[9]

There is another numerical method that is analytically more complex, but it is simple to use and has a benefit worth presenting. Isaac Newton invented an iterative procedure using calculus to provide numerical solutions to such vexing equations. His procedure involves making an initial guess of the solution, then subtracting the equation itself divided by its own first derivative to provide a second guess. We repeat the process until we converge to a single answer.

The benefit of Newton's method is that it will enable us to simply enter assumptions for the cash flow base and the perpetual growth, and the spreadsheet will automatically calculate the value of the firm without our having to manually go through the iterations as we did in Tables IV-A, B, and C. Remember, some iteration process is necessary when using log size discount rates, because the discount rate is not independent of size, as it is using other discount rate models.

To use Newton's procedure, we rewrite [8] as:

$$[9] \qquad \text{Let } f(V) \quad = \quad V - [CF / (a + b \ln V - g)] = 0$$

$$[10] \qquad f'(V) \quad = 1 + [(b\, CF) / V (a + b \ln V - g)^2]$$

Assuming our initial guess of value is V_0, the formula that defines our next iteration of value, V_1, is:

$$[11] \qquad V_1 = \quad V_0 - \frac{V - [CF / (a + b \ln V_0 - g)]}{1 + [(b\, CF) / V_0 (a + b \ln V_0 - g)^2]}$$

Table V shows Newton's Iterative Process for the simplest valuation. In Rows 18-22 we enter our initial guess of value of an arbitrary $26,000, our forecast cash flow base of $100,000, perpetual growth of 7%, and our regression coefficients a and b.

[8] I thank my friend William Scott, Jr., a physicist, for the terminology and the definite statement that there is no analytic solution.

[9] This is an optional section of the article and is the most mathematically difficult section.

In Row 3, we see our initial guess of $26,000. Row 4, with the iteration #2 value of $13,626,578 contains the formula in Row 34, which is equation [11] in a Lotus 123 formula format.[10] Rows 5-8 are simply the formula in Row 4 copied to the remaining spreadsheet cells.

Once we have the formula, we can value any firm with constant growth in its cash flows by simply changing the parameters in Rows 19-20.

Rows 27-30 show the sensitivity of the model to the initial guess. If we guess badly enough, the model will explode instead of converging to the right answer. For this particular set of assumptions, an initial guess of anywhere between $26,000 and $73 billion will converge to the right answer. Above $74 billion or below $25,000 explodes the model.

Unfortunately, the midyear Gordon Model, which is more accurate, has a much more complex formula. The iterative process does converge, but much too slowly to be of any practical use. When Lotus 123 and Excel will provide spreadsheets with 10 million rows, perhaps I will provide the formula. However, I doubt Business Valuation Review will print the spreadsheet!

VI. Measuring Valuation Error

It is a useful question to ask how accurate we are in valuation. It is virtually impossible to say how accurate we are in forecasting cash flows, and as a profession, we have little idea how accurate we are in calculating discounts for lack of marketability and minority interest or control premiums. However, we can at least place some reliable estimates on the accuracy of our discount rates and see how that impacts value.

We use a midyear Gordon Model formula as our valuation formula. The results we obtain are obviously inapplicable to startups, as the Gordon Model presupposes that the company being valued has constant perpetual growth.

Table VI: 95% Confidence Intervals

Table VI contains calculations of 95% confidence intervals around the valuation that results from our calculation of discount rate. For purposes of this exercise, we will assume the forecast cash flows and perpetual growth rate are correct, so we can see the impact of the statistical uncertainty of the discount rate.

[10] Cell C7, our initial guess, is V_0 in the equation

Column 2: Valuing The Mega-Firm

In Row 1, we show last year's cash flows as $300 million. The discount rate in Row 2 is 11%, and the perpetual growth rate in Row 3 is 9%. In Row 4, we apply the perpetual growth rate to calculate forecast cash flows for the first forecast year, which is $327 million.

In Row 8, we repeat the 11% discount rate. In Rows 7 and 9, we form a 95% confidence interval around the 11% rate. Regression #2 has 10 observations and 8 degrees of freedom. Using a t-distribution with 8 degrees of freedom, we must add and subtract 2.306 standard errors to form a 95% confidence interval. The standard error of the Abrams equation is 0.89%, which when we multiply that by 2.306 = 2.05%. The upper bound of the discount rate calculated by Regression Equation #2 is then 11% + 2.05% = 13.05% (Row 7), and the lower bound is 11% - 2.05% = 8.95% (Row 9).

In Row 12, we assume that somehow magically CAPM will come to the same 11% discount rate. We multiply the CAPM standard error of 2.72% (Table I, Column 12) x 2.306 standard errors = 6.27% for our 95% confidence interval. We add 6.27% in Row 11 and subtract it in Row 13 to arrive at upper and lower bounds of 17.27% and 4.73%, respectively.

In Row 16, we calculate a Gordon Midyear Multiple, i.e., GMM = $(1+r)^{.5}/(r-g)$, where r = 11%, as per Row 12. This comes to 52.6783, which we multiply by the $327 million cash flow in Row 4 to come to a FMV (ignoring discounts and premiums) of $17.226 billion in Row 20.

We repeat that process using 13.05%, the upper bound of the 95% confidence interval for the discount rate (Row 7) in the GMM formula to come to a lower bound of the GMM of 26.2382 (Row 15). Similarly, we use the lower bound of 8.95% (Row 9) in the GMM formula to calculate the upper bound of the GMM. However, the model explodes, as the discount rate of 8.95% is less than the growth rate. Those results appear in Row 17.

The FMVs associated with the GMMs are $8.58 billion (Row 19) and the nonsense value of -$652 billion in Row 20 for the exploded multiple.

In Row 19, Column 3, we show the lower bound is 49.8% of our best estimate of $17.226 billion using the Abrams Model. The upper bound is meaningless.

In Row 24, we show the Gordon Model Multiple for CAPM, which by assumption is the same as the Abrams Model. In Rows 23 and 25, we show the lower and upper bounds of the 95% confidence interval for the Gordon Model, which are 14.5037 and -29.8670, respectively. Obviously, the latter is an explosive result.

Row 28 shows the same estimate of FMV for CAPM as the Abrams Model, but look at the lower bound estimate in Row 27. It is $4.7 billion, or 27.5% of the best estimate, versus

49.8% for the same in the Abrams Model. The CAPM standard error being more than three times larger creates a huge confidence interval and often leads to explosive results for large firms.

Column 4: Using the Decile #1-#9 Regression

In Column 4, we are valuing the same firm as in Column 2. The only difference is that we use the regression parameters from the third column in Table I, where we exclude decile #10 (the smallest firms) as a potential outlier. R^2 rises to 98.83%, and Adjusted R^2 is 98.67%. That is even higher than results in Regression #1. The standard error of the y estimate is only 0.23%, about ¼ of the error with decile #10 included. It appears that this regression is appropriate for valuing firms that are at least as large as decile #9, or about $100 million in value. It is inappropriate to exclude decile #10, the smallest firms, and use that regression to extrapolate the discount rate to firms even smaller than decile #10 firms.

Row 42 shows the 0.23% standard error, which we use only in Column 4. In all other columns, we use the 0.89% standard error for the Abrams Equation in Row 41.

Now we have only 9 observations and 7 degrees of freedom. To form a 95% confidence interval, we need 2.365 standard errors on either side of our estimate. 2.365 x .23% = 0.54%, which we add and subtract from Row 8 to come to the lower and upper GMMs in Rows 7 and 9 of 11.54% and 10.46%. This is a much tighter estimate. The lower and upper bound FMVs are $13.6 billion in Row 19 and $23.6 billion in Row 21, which are 21.2% below and 37.0% above the $17.226 billion FMV. In Row 35, we average the 21.2% and 37.0% errors to come to an average 29.1% error of the Abrams Model to form a 95% confidence interval for the Mega-Firm.

Column 6: Valuation Error of a $100 Million Firm

In Column 6, we assume the firm had cash flows of $15 million last year and will grow at 7%. We see in Rows 35-36 that the Abrams Model has an average error of 18.4% and CAPM has an average error of 62.4% in forming a 95% confidence interval.

Columns 8 and 10 are successively smaller firms. Note how the valuation error declines with firm size.

Summary of Valuation Errors From Statistical Uncertainty in the Discount Rate

Using the Abrams Model, i.e., Regression Equation #2, we see that under the best of circumstances, megafirms can easily have a 30% valuation error arising just from the statistical uncertainty in calculating the discount rate. The huge firms have larger errors because we are "closer to the edge," i.e., where the growth rate approaches the discount rate.[11] Small to medium firms have valuation errors of 9%-10% in forming a 95% confidence interval. In other words, we are 95% sure that the valuation error is no more than 9%-10%. Probably it is less than that, but we are 95% sure that is the outer limit.

CAPM, on the other hand, has very large valuation error: 24-30% for small to medium firms, and 60% to infinite errors for large firms.

It is worth noting that the assumption of a symmetric t distribution around the discount rate results in an asymmetric distribution of value, with a larger range of probable error on the high side than the low side.

When we add all the other sources of errors in valuation, it is indeed not surprising at all that professional appraisers can vary widely in their results.

Valuation Error From Other Sources: Comparative Static Analysis

We have focused on the errors in the discount rate as a source for valuation error. Because of the statistical nature of our derivation of the regression equations, we can be very precise statistically in forming 95% confidence intervals around our valuation estimate. We cannot do that with other sources of valuation error; however we can make some qualitative and quantitative observations using comparative static analysis common in economics.

Errors in Forecasting Cash Flow

We will repeat equation [6] as [12] for convenience, and then we will perform some calculus on [12] in order to see the effect of errors in calculating the cash flows and growth rates. We use the simplest end year Gordon Model, as the mathematics are difficult enough without the complications of the midyear correction.

[12] $V = CF / (r-g)$ The Gordon Model-End year assumption[12]

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[11] Smaller firms with very high expected growth will also be "close to the edge"

[12] For simplicity, we will stick to this simple equation and ignore the more proper $r = a + b \ln V - g$ for the remainder of this section

We take natural logs of both sides of the equation, as we will be using logarithmic derivatives in our analysis.

[13] $\ln V = \ln CF - \ln (r-g)$

Next we take the partial derivative of [12] with respect to cash flow (CF) to see the effect of changes in cash flow on the value of the firm.

[14] $$\frac{\partial V}{\partial CF} = \frac{1}{(r-g)}$$

We see that for a small change in cash flow, the value increases by 1/(r-g). In absolute dollars, large firms will experience a larger increase in value than small firms for each additional dollar of cash flow. The reason is that r is smaller for large firms. If g is the same—which it usually is not—then value increases more for the large firm for the same additional dollar of cash flow. This applies to valuation error. If we overestimate cash flows by $1, r = .11, and g = .09, then value increases by 1/(.11-.09) = 1/.02 = $50. You can see that in Table VII, Row 3, Column 2. For the small firm in Column 3, r = .27 and g=.05, so 1/(r-g) = 1/.22 = $4.55 (Row 3).

Here again we find that larger firms will tend to have larger valuation error—at least in absolute dollars. Let's look at the same question in percentage terms. We can calculate the percentage error by taking the logarithmic derivative, which gives us the instantaneous rate of change at the margin.[13]

[15] $$\frac{\partial \ln V}{\partial CF} = \frac{1}{CF}$$

The instantaneous rate of change for each $1 error in forecasting cash flows is higher for small firms, because their denominator is smaller than that of large firms.

A k% error in forecasting cash flows for both a large firm and a small firm increases value in both cases by k%, as we can see in [16] - [18] below.

[16] $$V_1 = \frac{CF}{(r-g)}$$

[17] $$V_2 = \frac{(1+k)CF}{(r-g)} = (1+k)\, V_1$$

[13] You can also obtain the same result by dividing [14] by [12]

[18] $\%\triangle V = (V_2 / V_1) - 1 = k$, i.e., there is a k% change in value regardless of the initial firm size.

Errors in Forecasting Growth

Next, let's evaluate the effects of an error in forecasting the growth rate. We take the partial derivative of V with respect to g.

[19] $$\frac{\partial V}{\partial g} = \frac{CF}{(r-g)^2} = \frac{-\partial V}{\partial (r-g)} = -\frac{\partial V}{\partial r}$$

Changes (or mistakes) in g, the growth rate, have a major impact on value. The squared term in the denominator is going to be very small, which means the impact on value will be very large—much larger than mistakes in forecasting cash flow. Comparing the partial derivatives in [14] and [19], the latter is much larger, because the numerator is CF vs. 1, and the denominator is $(r-g)^2$ vs. (r-g). *This means we need to pay relatively more attention to forecasting growth rates and discount rates than we do to forecasting cash flows, and the larger the firm, the more painstaking should be the analysis.*

[19] shows that mistakes in forecasting r or g will have a much larger impact on large firms than small firms. Changes in r, the discount rate, have the identical effect on value as changes in g, but in the opposite direction. Overestimating growth increases value, while overestimating the discount rate decreases value by the same amount. The middle partial differential in [19] tells us that it is the net effect of r-g that is ultimately significant in the impact of errors in either r, g, or both on value.

We can investigate the impact of a k% error in estimating g in the following manner:

[20] $V_2 = CF / [r - (1+k)g]$

[21] $V_2 / V_1 = (r - g) / [r - (1+k) g]$

The percentage error in value resulting from a *percentage* error in forecasting growth (in [19] we were measuring the valuation error resulting from an *absolute* error in forecasting growth) will be $[V_2/V_1)-1]$, i.e., the ratio in [21] minus one.

Commentary: Table VII

Table VII shows the calculations for k = 10% error in forecasting growth. In Rows 1-2, we show the discount rate and growth rate for a large firm in Column 2 and a small firm in Column 3, respectively. The End of Year Gordon Model multiples in Row 3 are 50 and 4.5455 for the large and the small firm. Multiplying that by the forecast cash flows in Row 4, we come to values in Row 5 of $15 billion and $454,545, respectively.

Now let's see what happens if we forecast growth too high by 10% for each firm. Row 6 shows the overly high growth rates of 9.9% and 5.5%. Row 6 contains the new Gordon Model multiples, and Row 8 shows the incorrect values we obtain with the high growth. Row 9 shows the ratio of the incorrect forecast to the correct forecast, i.e., V_2/V_1, and Row 10 shows the percentage error of 81.82% for the large firm and 2.33% for the small firm.

The Sensitivity Analysis in Rows 15-31 shows the percentage valuation error for various combinations of the original r and g using the right hand side of [21] minus one as the formula for the error, with k=10%. Note that the shaded cells in Rows 15 and 31 match the results in Row 10, which confirms the accuracy of the error formula.

Again, we find that equal percentage errors in forecasting growth will create much larger valuation errors for large firms than small firms. In particular, the valuation error is very sensitive to the growth rate and somewhat less sensitive to the discount rate. It is sensitive to both, however. Again, r-g is very important in the magnitude of valuation error, more so than r or g by themselves.

Another issue in valuation error is that while a sole initial error in calculating the discount rate is self-correcting by using an iterative method, an error in calculating cash flows or the growth rate not only causes its own error, but also will distort the calculation of the discount rate. For example, overestimating growth, g, will cause an overvaluation, which will lower the discount rate beyond its proper level, which will in turn cause a second order over- valuation. We did not see this in our comparative static analysis, because for simplicity we were working with a simpler Gordon Multiple in the form of [16]. This secondary valuation error caused by a faulty forecast of cash flows or growth rate will be minimal, because the discount rate is quite insensitive to the error in the estimate of value.

VII. Summary & Conclusions

The Abrams Model is not only far more accurate than CAPM for valuing privately-held businesses, but it is much faster and easier to use. It requires no research on your part. CAPM often requires 1-2 days research of the "comparables" (Guideline Companies).

Speaking of "comparables," it is very inaccurate to apply the betas for IBM, Compaq, Apple Computer, etc. to a small startup computer firm with $2 million in sales. The size effect drowns out any real information in betas, especially when applied to small firms. The threefold improvement in standard error in the Abrams Equation vs. CAPM applies only to firms of the same magnitude. When applied to small firms, CAPM yields even more erroneous results, unless the appraiser compensates by blindly adding another 5%-10% beyond the typical Ibbotson "Small Firm Premium" and calling that a Specific Company Adjustment (SCA). I suspect this practice is common, but then it is not really an SCA; rather it is an intuitive attempt to compensate for a model that has no place being used to value small and medium firms.

In a recent valuation of a mid-size firm with $25 million in sales, $2 million in net income after taxes, and very fast growth, I used a Guideline Company Approach, among others. I found 16 Guideline Companies with positive earnings in the same SIC Code. I regressed the value of the firm against net income, with "great" results—99.5% R^2 and high t statistics. When I applied the regression equation to my Subject Company, the value came to -$91 million![14] I suspect that much of this goes on with CAPM as well. I would bet that many appraisers seriously overvalue small companies using discount rates appropriate for large firms only.

When using the Abrams Model, we extrapolate the discount rate to the appropriate level for each firm that we value. There is no further need for a size adjustment. We merely need to compare our Subject Company to other companies of its size, not to IBM. For that reason, using Robert Morris Associates to compare the Subject Company to other firms of its size is very important, as that is truly comparable.

Since we have already extrapolated the rate of return through the regression equation in a manner that appropriately considers the average risk of being any particular size, the relevant comparison when considering Specific Company Adjustments is to other companies of its same size. There is a difference between two firms that each do $2 million in sales volume when one is a one-man show and the other firm has two Harvard MBAs running it. If the former is closer to average management, you should probably subtract 1% or 2% from the discount rate for the latter; if the latter is the norm, it is appropriate to add that much to the discount rate of the former.

[14] I solved the magnitude problem by regressing ln Value against ln Net Income. That eliminated the scaling problem and led to reasonable results.

Table I

NYSE Data By Decile & Statistical Analysis: 1926-1995

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	Y	X1					X2
			Recent Mkt				
Decile	Arith Mean Return	Std Dev	Capitalization	% Cap	# Co.s	Avg Cap=FMV	Ln(FMV)
1	**11.42%**	**18.95%**	3,110,306,745,000	64.23%	169	18,404,181,923	**23.6358**
2	**13.36%**	**22.55%**	743,402,451,000	15.35%	169	4,398,831,071	**22.2046**
3	**14.07%**	**24.39%**	384,020,909,000	7.93%	170	2,258,946,524	**21.5382**
4	**14.65%**	**26.84%**	226,702,002,000	4.68%	169	1,341,431,964	**21.0170**
5	**15.60%**	**27.67%**	146,129,715,000	3.02%	169	864,672,870	**20.5779**
6	**15.53%**	**28.72%**	98,979,665,000	2.04%	169	585,678,491	**20.1883**
7	**15.98%**	**31.18%**	64,087,771,000	1.32%	170	376,986,888	**19.7477**
8	**17.11%**	**35.01%**	39,063,761,000	0.81%	169	231,146,515	**19.2586**
9	**17.86%**	**37.56%**	21,589,252,000	0.45%	169	127,747,053	**18.6656**
10	**22.04%**	**46.81%**	8,220,123,000	0.17%	170	48,353,665	**17.6941**
Std Deviation	**2.88%**				**1,693**		
Value Wtd Index	11.86%	20.24%	**4,842,502,394,000**	**100.00%**			

1st Regression: Return = F(Std Dev. of Returns)

	Deciles 1-10		#1-9 Only
	1926-1995	1926-1993	1926-1995
Constant	5.24%	4.98%	5.89%
70/68 Year CAGR LT Bonds	5.17%	5.02%	5.17%
Std Err of Y Est	0.43%	0.47%	0.41%
R Squared	98.02%	97.87%	96.24%
Adjusted R Squared	97.78%	97.56%	95.70%
No. of Observations	10	10	9
Degrees of Freedom	8	8	7
X Coefficient(s)	35.11%	35.86%	32.64%
Std Err of Coef.	1.76%	1.87%	2.44%
T	19.9	19.2	13.4
P	< .01%	< .01%	< .01%

2nd Regression: Return = F[LN(Mkt Capitalization)]

	Deciles 1-10		#1-9 Only
	1926-1995	1926-1993	1926-1995
Constant	47.94%	49.27%	0.41%
Std Err of Y Est	0.89%	0.77%	0.23%
R Squared	91.43%	94.20%	98.83%
Adjusted R Squared	90.36%	93.46%	98.67%
No. of Observations	10	10	9
Degrees of Freedom	8	8	7
X Coefficient(s)	-1.57334352%	-1.63%	-1.26165%
Std Err of Coef.	0.17%	0.14%	0.05184%
T	-9.2	-11.6	-24.3
P	< .01%	< .01%	< .01%

3rd Regression: Return = F[Decile Beta]

	1926-1995
Constant	-4.60%
Std Err of Y Est	0.70%
R Squared	94.76%
Adjusted R Squared	94.11%
No. of Observations	10
Degrees of Freedom	8
X Coefficient(s)	17.14%
Std Err of Coef.	1.42%
T	12.0
P	< .01%

[1] Regression #3 Std Errors, R-Squares are identical to Regression #1

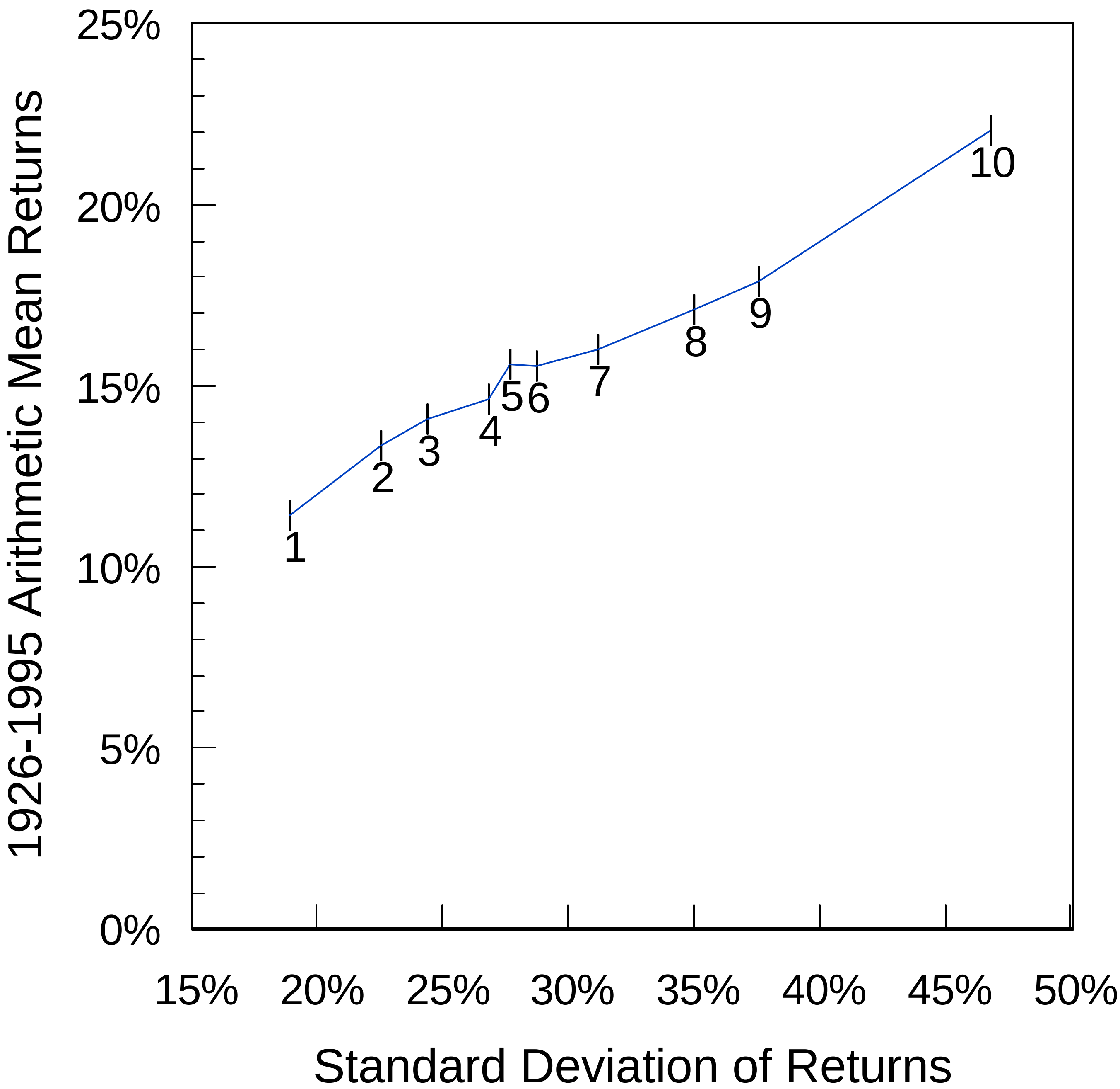
Table I (cont'd)

Comparing Abrams' Model To CAPM

(9)	(10)	(11)	(12)	(13)	(14)	(15)
Note [1]	Note [2]	=(2) - (10)	=(11)^2		=(2) - (13)	=(14)^2
		CAPM		**Regr #2**	**Regr #2**	
Beta	**CAPM E(R)**	**Error**	**Sq Error**	**Estimate**	**Error**	**Sq Error**
0.90	11.83%	-0.41%	0.0017%	10.75%	0.67%	0.0044%
1.04	12.86%	0.50%	0.0025%	13.01%	0.35%	0.0013%
1.09	13.23%	0.84%	0.0070%	14.05%	0.02%	0.0000%
1.13	13.53%	1.12%	0.0125%	14.87%	-0.22%	0.0005%
1.17	13.83%	1.77%	0.0314%	15.57%	0.03%	0.0000%
1.19	13.97%	1.56%	0.0242%	16.18%	-0.65%	0.0042%
1.24	14.34%	1.64%	0.0267%	16.87%	-0.89%	0.0079%
1.29	14.71%	2.40%	0.0574%	17.64%	-0.53%	0.0028%
1.36	15.23%	2.63%	0.0690%	18.57%	-0.71%	0.0051%
1.47	16.05%	5.99%	0.3592%	20.10%	1.94%	0.0375%
	Totals ----------->		**0.5916%**			**0.0638%**
	Standard Error ->		**2.72%**			**0.89%**
	Std Error-CAPM / Std Error-Abrams					**304.49%**

Fig. 1

1926-1995 Mean Returns as a Function of Std. Deviation

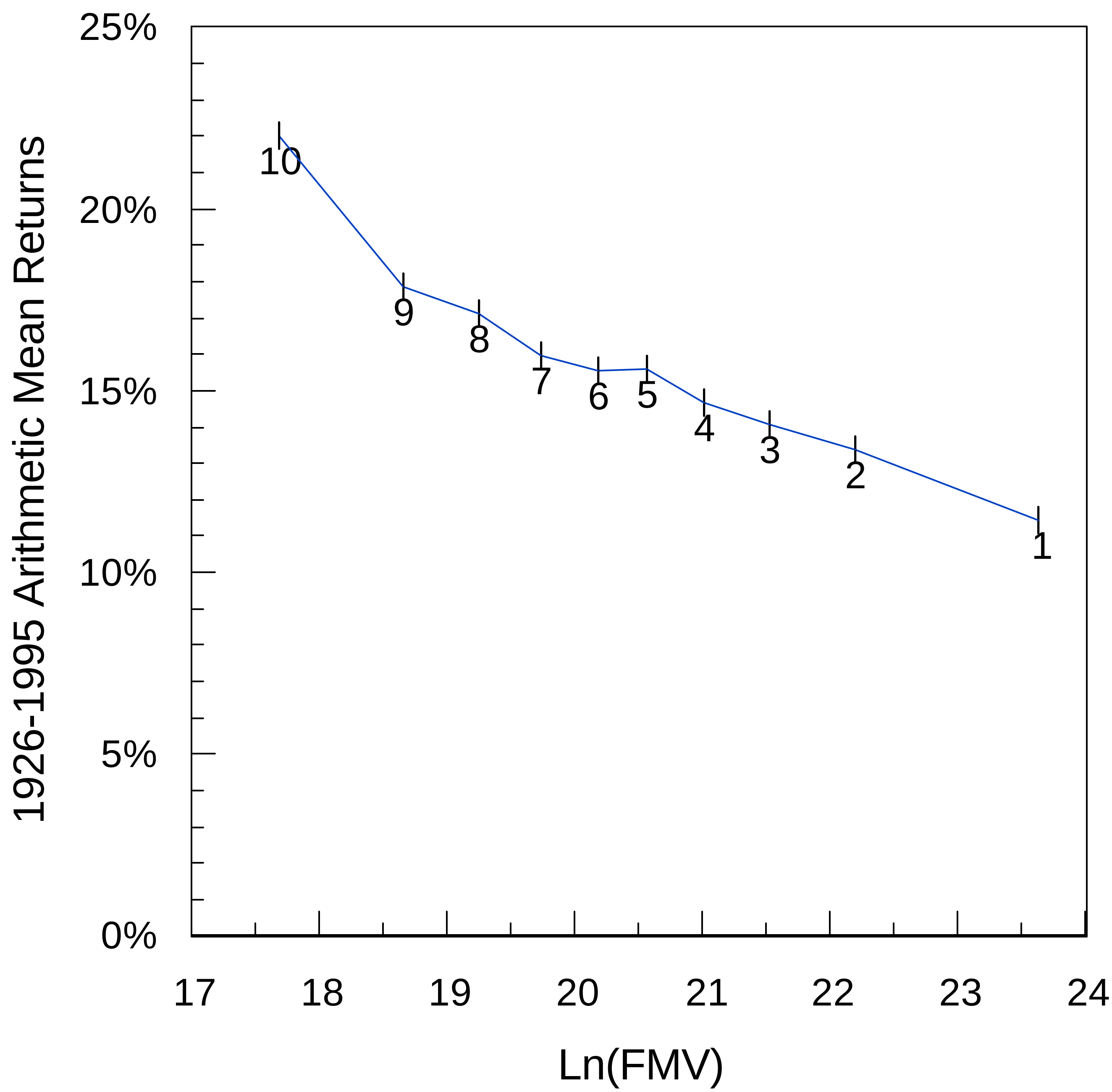


These are arithmetic mean returns for the CRSP deciles. D

Regression #1: r = 5.24% + (.3511 * Std Dev of Decile)

Fig. 2

1926-1995 Mean Returns as a Function of Ln(FMV)



These are arithmetic mean returns for the CRSP deciles. Data labels
Regression #2: r = 47.94% - [.0157334352 * Ln(FMV)]

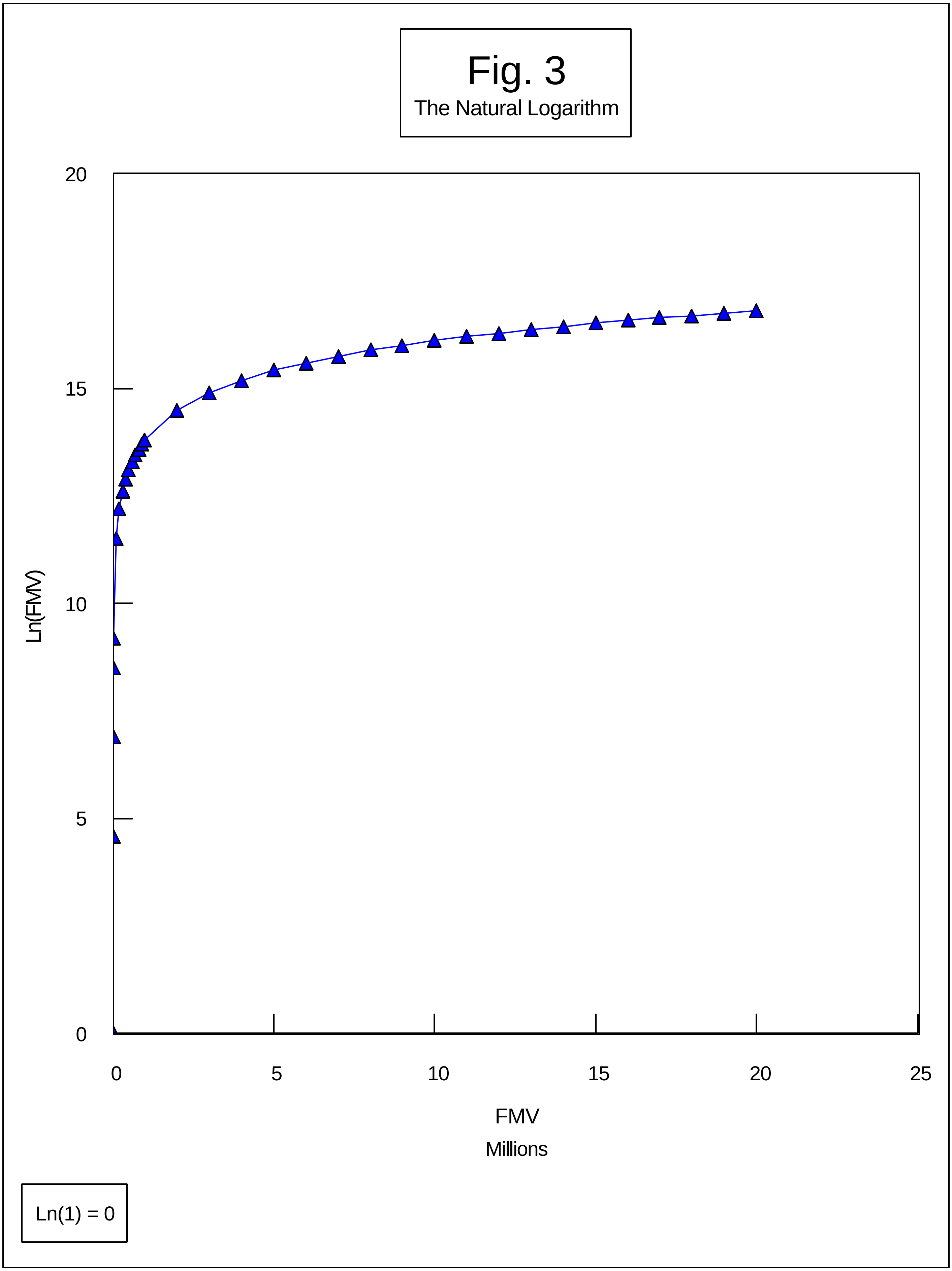


Fig. 3
The Natural Logarithm
20
15
10
5
0
Ln(FMV)
0
5
10
15
20
25
FMV
Millions
Ln(1) = 0

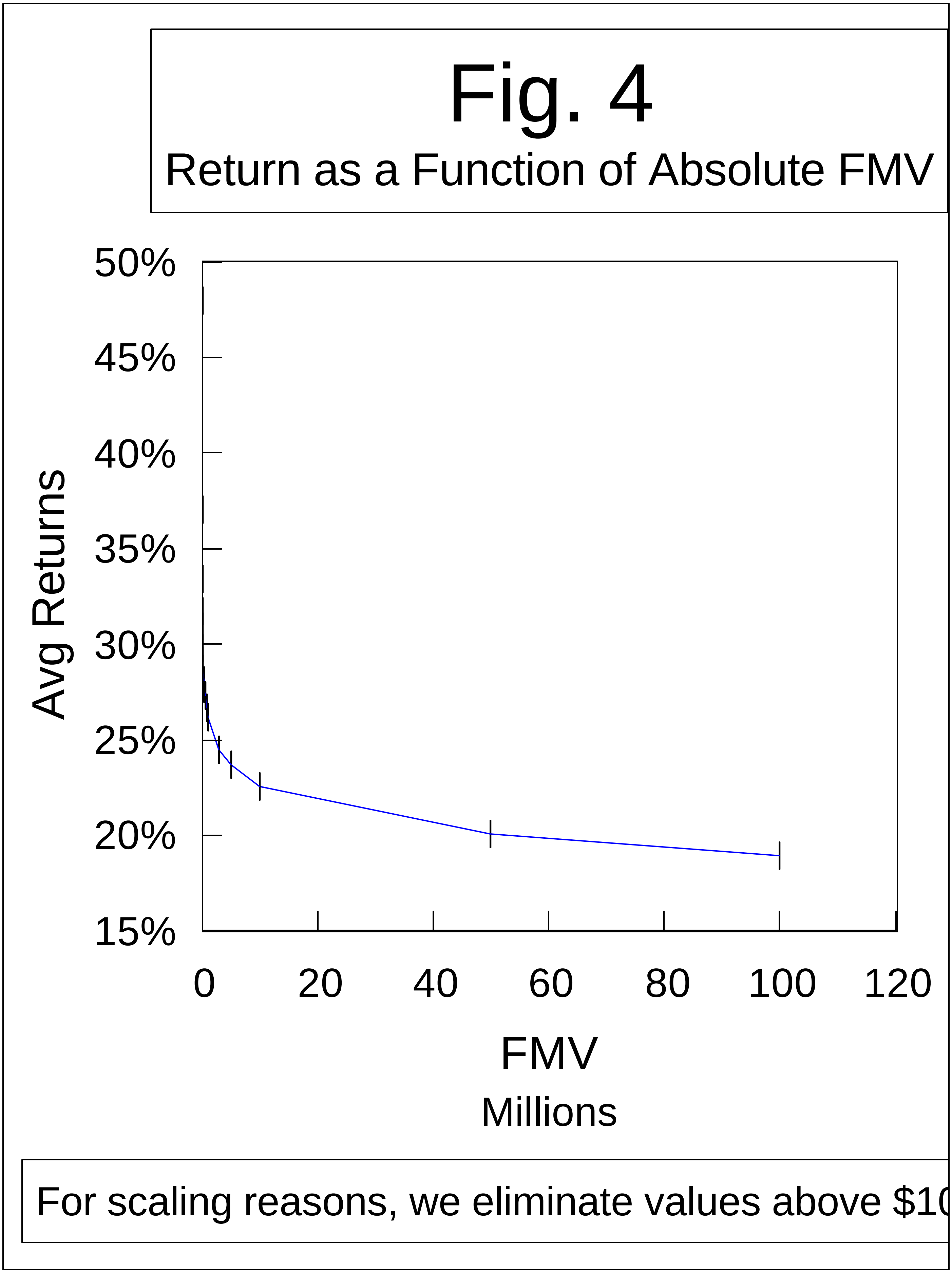


Fig. 4
Return as a Function of Absolute FMV
Avg Returns
50%
45%
40%
35%
30%
25%
20%
15%
0
20
40
60
80
100
120
FMV
Millions
For scaling reasons, we eliminate values above $10

Table II

Abrams Table of Stock Market Returns Based on FMV

Regression Results	Implied Discount
Mktable Min FMV	Rate (R)
$10,000,000,000	11.7%
$1,000,000,000	15.3%
$100,000,000	19.0%
$50,000,000	20.0%
$10,000,000	22.6%
$5,000,000	23.7%
$3,000,000	24.5%
$1,000,000	26.2%
$750,000	26.7%
$500,000	27.3%
$400,000	27.6%
$300,000	28.1%
$200,000	28.7%
$150,000	29.2%
$100,000	29.8%
$50,000	30.9%
$30,000	31.7%
$10,000	33.5%
$1,000	37.1%
$1	47.9%

Assumptions:

Long-Term Gov't Bonds: Total Return Index [1]	12/31/1995	34.044
Long-Term Gov't Bonds: Total Return Index [1]	12/31/1925	1.000
70 Yr Compound Return = 34.044^(1/70) - 1	70 Yr Return	5.17%
Long Horizon Equity Premium [2]		7.4%

Notes:

[1] (Source: SBBI-1996, p. 240-241)
[2] SBBI-1996 p. 161

Table II

NYSE Data By Decile & Statistical Analysis

(1)	(2)	(3)	(4)	(5)	(6) =(5)/158 Firms	(7)
	Y	X1	Recent Mkt			X2
	Geometric		Capitali-	Decile	Firm	
Decile	Mean	Std Dev	zation	Capitalization	Capitalization	Ln(FMV)
1	9.42%	18.89%	61.98%	$2,733,565,920,000	$17,301,050,127	23.5740
2	10.83%	22.66%	16.11%	710,515,440,000	4,496,933,165	22.2267
3	11.37%	24.60%	8.45%	372,678,800,000	2,358,726,582	21.5814
4	11.44%	27.10%	5.06%	223,166,240,000	1,412,444,557	21.0686
5	12.14%	27.94%	3.43%	151,276,720,000	957,447,595	20.6798
6	11.78%	29.08%	2.13%	93,941,520,000	594,566,582	20.2033
7	11.73%	31.62%	1.42%	62,627,680,000	396,377,722	19.7979
8	11.82%	35.34%	0.87%	38,370,480,000	242,851,139	19.3080
9	12.20%	38.05%	0.42%	18,523,680,000	117,238,481	18.5797
10	13.84%	47.48%	0.14%	6,174,560,000	39,079,494	17.4811
Value Wtd Index	12.03%	20.34%	100.01%	$4,410,400,000,000		

Regression [1]: Return = F(Std Dev. of Returns)

Constant	4.98%
Std Err of Y Est	0.47%
R Squared	97.87%
No. of Observations	10
Degrees of Freedom	8
X Coefficient	35.86%
Std Err of Coef.	1.87%
T	19.2
P	<.01%

Regression [2]: Return = F[LN(Mkt Capitalization)]

	Geometric	Arithmetic
Constant	23.45%	49.27%
Std Err of Y Est	0.45%	0.77%
R Squared	85.60%	94.20%
No. of Observations	10	10
Degrees of Freedom	8	8
X Coefficient	-0.005765	-0.016350
Std Err of Coef.	0.000836	0.0014
T	-6.9	-11.4
P	< .01%	< .01%

Table III
Stock & Bond Returns

	Y	X		$Y-\bar{Y}$	$X-\bar{X}$	$(X-\bar{X})(Y-\bar{Y})$	Z	$Z-\bar{Z}$	$(Y-\bar{Y})(Z-\bar{Z})$
	Lge Co	LT T-Bond	Diff=				LT Treas		
	Stocks [1]	Return [2]	Eqty Prem				Bond Yld [3]		
1926	0.1162	0.0777	0.0385	-0.0090	0.0223	-0.000201	0.0354	-0.0171086	0.000154
1927	0.3749	0.0893	0.2856	0.2497	0.0339	0.008461	0.0316	-0.0209086	-0.005220
1928	0.4361	0.0010	0.4351	0.3109	-0.0544	-0.016915	0.0340	-0.0185086	-0.005754
1929	-0.0842	0.0342	-0.1184	-0.2094	-0.0212	0.004442	0.0340	-0.0185086	0.003876
1930	-0.2490	0.0466	-0.2956	-0.3742	-0.0088	0.003297	0.0330	-0.0195086	0.007301
1931	-0.4334	-0.0531	-0.3803	-0.5586	-0.1085	0.060617	0.0407	-0.0118086	0.006597
1932	-0.0819	0.1684	-0.2503	-0.2071	0.1130	-0.023403	0.0315	-0.0210086	0.004351
1933	0.5399	-0.0007	0.5406	0.4147	-0.0561	-0.023268	0.0336	-0.0189086	-0.007841
1934	-0.0144	0.1003	-0.1147	-0.1396	0.0449	-0.006268	0.0293	-0.0232086	0.003241
1935	0.4767	0.0498	0.4269	0.3515	-0.0056	-0.001972	0.0276	-0.0249086	-0.008755
1936	0.3392	0.0752	0.2640	0.2140	0.0198	0.004234	0.0255	-0.0270086	-0.005779
1937	-0.3503	0.0023	-0.3526	-0.4755	-0.0531	0.025256	0.0273	-0.0252086	0.011987
1938	0.3112	0.0553	0.2559	0.1860	-0.0001	-0.000021	0.0252	-0.0273086	-0.005079
1939	-0.0041	0.0594	-0.0635	-0.1293	0.0040	-0.000516	0.0226	-0.0299086	0.003868
1940	-0.0978	0.0609	-0.1587	-0.2230	0.0055	-0.001224	0.0194	-0.0331086	0.007384
1941	-0.1159	0.0093	-0.1252	-0.2411	-0.0461	0.011119	0.0204	-0.0321086	0.007742
1942	0.2034	0.0322	0.1712	0.0782	-0.0232	-0.001815	0.0246	-0.0279086	-0.002182
1943	0.2590	0.0208	0.2382	0.1338	-0.0346	-0.004630	0.0248	-0.0277086	-0.003707
1944	0.1975	0.0281	0.1694	0.0723	-0.0273	-0.001974	0.0246	-0.0279086	-0.002017
1945	0.3644	0.1073	0.2571	0.2392	0.0519	0.012410	0.0199	-0.0326086	-0.007799
1946	-0.0807	-0.0010	-0.0797	-0.2059	-0.0564	0.011617	0.0212	-0.0313086	0.006447
1947	0.0571	-0.0262	0.0833	-0.0681	-0.0816	0.005560	0.0243	-0.0282086	0.001922
1948	0.0550	0.0340	0.0210	-0.0702	-0.0214	0.001504	0.0237	-0.0288086	0.002023
1949	0.1879	0.0645	0.1234	0.0627	0.0091	0.000570	0.0209	-0.0316086	-0.001981
1950	0.3171	0.0006	0.3165	0.1919	-0.0548	-0.010517	0.0224	-0.0301086	-0.005777
1951	0.2402	-0.0393	0.2795	0.1150	-0.0947	-0.010889	0.0269	-0.0256086	-0.002944
1952	0.1837	0.0116	0.1721	0.0585	-0.0438	-0.002562	0.0279	-0.0246086	-0.001439
1953	-0.0099	0.0364	-0.0463	-0.1351	-0.0190	0.002569	0.0274	-0.0251086	0.003393
1954	0.5262	0.0719	0.4543	0.4010	0.0165	0.006611	0.0272	-0.0253086	-0.010148
1955	0.3156	-0.0129	0.3285	0.1904	-0.0683	-0.013005	0.0295	-0.0230086	-0.004380
1956	0.0656	-0.0559	0.1215	-0.0596	-0.1113	0.006637	0.0345	-0.0180086	0.001074
1957	-0.1078	0.0746	-0.1824	-0.2330	0.0192	-0.004471	0.0323	-0.0202086	0.004709
1958	0.4336	-0.0609	0.4945	0.3084	-0.1163	-0.035867	0.0382	-0.0143086	-0.004412
1959	0.1196	-0.0226	0.1422	-0.0056	-0.0780	0.000439	0.0447	-0.0078086	0.000044
1960	0.0047	0.1378	-0.1331	-0.1205	0.0824	-0.009930	0.0380	-0.0145086	0.001749
1961	0.2689	0.0097	0.2592	0.1437	-0.0457	-0.006568	0.0415	-0.0110086	-0.001582
1962	-0.0873	0.0689	-0.1562	-0.2125	0.0135	-0.002867	0.0395	-0.0130086	0.002765
1963	0.2280	0.0121	0.2159	0.1028	-0.0433	-0.004451	0.0417	-0.0108086	-0.001111
1964	0.1648	0.0351	0.1297	0.0396	-0.0203	-0.000804	0.0423	-0.0102086	-0.000404
1965	0.1245	0.0071	0.1174	-0.0007	-0.0483	0.000035	0.0450	-0.0075086	0.000005
1966	-0.1006	0.0365	-0.1371	-0.2258	-0.0189	0.004271	0.0455	-0.0070086	0.001583
1967	0.2398	-0.0918	0.3316	0.1146	-0.1472	-0.016867	0.0556	0.00309143	0.000354
1968	0.1106	-0.0026	0.1132	-0.0146	-0.0580	0.000848	0.0598	0.00729143	-0.000107
1969	-0.0850	-0.0507	-0.0343	-0.2102	-0.1061	0.022307	0.0687	0.01619143	-0.003404
1970	0.0401	0.1211	-0.0810	-0.0851	0.0657	-0.005592	0.0648	0.01229143	-0.001046
1971	0.1431	0.1323	0.0108	0.0179	0.0769	0.001374	0.0597	0.00719143	0.000129
1972	0.1898	0.0569	0.1329	0.0646	0.0015	0.000096	0.0599	0.00739143	0.000477
1973	-0.1466	-0.0111	-0.1355	-0.2718	-0.0665	0.018080	0.0726	0.02009143	-0.005461
1974	-0.2647	0.0435	-0.3082	-0.3899	-0.0119	0.004645	0.0760	0.02349143	-0.009160
1975	0.3720	0.0920	0.2800	0.2468	0.0366	0.009029	0.0805	0.02799143	0.006908
1976	0.2384	0.1675	0.0709	0.1132	0.1121	0.012686	0.0721	0.01959143	0.002217
1977	-0.0718	-0.0069	-0.0649	-0.1970	-0.0623	0.012277	0.0803	0.02779143	-0.005476
1978	0.0656	-0.0118	0.0774	-0.0596	-0.0672	0.004008	0.0898	0.03729143	-0.002224
1979	0.1844	-0.0123	0.1967	0.0592	-0.0677	-0.004007	0.1012	0.04869143	0.002881
1980	0.3242	-0.0395	0.3637	0.1990	-0.0949	-0.018885	0.1199	0.06739143	0.013409
1981	-0.0491	0.0186	-0.0677	-0.1743	-0.0368	0.006417	0.1334	0.08089143	-0.014101
1982	0.2141	0.4036	-0.1895	0.0889	0.3482	0.030945	0.1095	0.05699143	0.005065
1983	0.2251	0.0065	0.2186	0.0999	-0.0489	-0.004885	0.1197	0.06719143	0.006711
1984	0.0627	0.1548	-0.0921	-0.0625	0.0994	-0.006214	0.1170	0.06449143	-0.004032
1985	0.3216	0.3097	0.0119	0.1964	0.2543	0.049936	0.0956	0.04309143	0.008462
1986	0.1847	0.2453	-0.0606	0.0595	0.1899	0.011293	0.0789	0.02639143	0.001570
1987	0.0523	-0.0271	0.0794	-0.0729	-0.0825	0.006017	0.0920	0.03949143	-0.002880
1988	0.1681	0.0967	0.0714	0.0429	0.0413	0.001770	0.0918	0.03929143	0.001685
1989	0.3149	0.1811	0.1338	0.1897	0.1257	0.023840	0.0816	0.02909143	0.005518
1990	-0.0317	0.0618	-0.0935	-0.1569	0.0064	-0.001003	0.0844	0.03189143	-0.005005
1991	0.3055	0.1930	0.1125	0.1803	0.1376	0.024804	0.0730	0.02049143	0.003694
1992	0.0767	0.0805	-0.0038	-0.0485	0.0251	-0.001217	0.0726	0.02009143	-0.000975
1993	0.0999	0.1824	-0.0825	-0.0253	0.1270	-0.003216	0.0654	0.01289143	-0.000326
1994	0.0131	-0.0777	0.0908	-0.1121	-0.1331	0.014925	0.0799	0.02739143	-0.003071
1995	0.3743	0.3167	0.0576	0.2491	0.2613	0.065080	0.0603	0.00779143	0.001941
Avg	**0.1252**	**0.0554**	**0.0698**	**-0.0000**	**-0.0000**	**0.0035**	**0.0525**	**-0.0000**	**-0.0000**
Std Dev	**0.2042**	**0.0925**					**0.0301**		

0.003536	**Cov(X,Y)**
0.187247	**Corr(X,Y)**

-0.000034	**Cov(Y,Z)**
-0.005535	**Corr(Y,Z)**

[1] SBBI-1996, Exhibit A-1, Pages 180-181
[2] SBBI-1996, Exhibit A-6, Pages 190-191
[3] SBBI-1996, Exhibit A-9, Pages 196-197

Corr(X,Y) = Cov(X,Y) / (Std Err of X * Std Err of Y)

Table IV-A

Discounted Cash Flow Analysis Using Abrams' Formula -1st Iteration

Row	Description:	1996	1997	1998	1999	2000	Total
1	**Assumptions:**						
2	Base Adjusted Cash Flow	$100,000					
3	Growth Rate in Adj Cash Flow	12%	10%	9%	8%	7%	
4	Discount Rate = R	30%					
5	Growth Rate To Perpetuity=G	6%					
6	Control Premium	35%					
7	Discount-Lack of Marketability	40%					
8							
9	**5 Year Forecasts**						
10							
11	Forecast Cash Flow	$112,000	$123,200	$134,288	$145,031	$155,183	
12	Present Value Factor	0.8771	0.6747	0.5190	0.3992	0.3071	
13	PV of Cash Flow	**$98,230**	**$83,118**	**$69,691**	**$57,897**	**$47,654**	**$356,591**
14							
15	**Calculation of Fair Market Value:**						
16			**Formula**				
17	Forecast Cash Flow 2001	$164,494	Row 11 for 2000 X (1+ Row 5)				
18	Gordon Model Cap Rate	4.7507	SQRT (1+R) / (R-G)				
19	FMV 1999-Infinity as of 1/1/2001	$781,468	Row 17 X Row 18				
20	Present Value Factor-5 Yrs	0.2693	1/(1+R)^5 [Where 5 is # yrs from 1/1/96 to 1/1/2001]				
21	PV of 2001-Infinity Cash Flow	$210,472	Row 19 X Row 20				
22	Add PV of 1996-2000 Cash Flow	356,591	Total of Row 13				
23	FMV-Marketable Minority	$567,063	Row 21 + Row 22				
24	Control Premium	198,472	Row 6 X Row 23				
25	FMV-Marketable Control Interest	765,536	Row 23 + Row 24				
26	Disc-Lack of Marketability	(306,214)	- Row 7 X Row 25				
27	Fair Market Value	**$459,321**	Row 25 + Row 26				
28	**Calc of Disc Rate-Regr Eq #2**						
29	Ln (FMV-Marketable Minority)	13.2482	Ln(Row 23)				
30	* X Coefficient of -.0157334352	-0.2084	Row 29 * X Coefficient-Regr #2				
31	Constant	0.4794	Constant-Regression #2				
32	Discount Rate (Rounded)	**27%**	Row 30 + Row 31				

Table IV-B

Discounted Cash Flow Analysis Using Abrams' Formula-2nd Iteration

Row	Description:	1996	1997	1998	1999	2000	Total
1	**Assumptions:**						
2	Base Adjusted Cash Flow	$100,000					
3	Growth Rate in Adj Cash Flow	12%	10%	9%	8%	7%	
4	Disc Rate = R (Table IV-A, Row 32)	27%					
5	Growth Rate To Perpetuity=G	6%					
6	Control Premium	35%					
7	Discount-Lack of Marketability	40%					
8							
9	**5 Year Forecasts**						
10							
11	Forecast Cash Flow	$112,000	$123,200	$134,288	$145,031	$155,183	
12	Present Value Factor	0.8874	0.6987	0.5502	0.4332	0.3411	
13	PV of Cash Flow	**$99,384**	**$86,081**	**$73,880**	**$62,827**	**$52,933**	**$375,105**
14							
15	**Calculation of Fair Market Value:**						
16			**Formula**				
17	Forecast Cash Flow 2001	$164,494	Row 11 for 2000 X (1+ Row 5)				
18	Gordon Model Cap Rate	5.3664	SQRT (1+R) / (R-G)				
19	FMV 1999-Infinity as of 1/1/2001	$882,741	Row 17 X Row 18				
20	Present Value Factor-5 Yrs	0.3027	1/(1+R)^5 [Where 5 is # yrs from 1/1/96 to 1/1/2001]				
21	PV of 2001-Infinity Cash Flow	$267,187	Row 19 X Row 20				
22	Add PV of 1996-2000 Cash Flow	375,105	Total of Row 13				
23	FMV-Marketable Minority	$642,292	Row 21 + Row 22				
24	Control Premium	224,802	Row 6 X Row 23				
25	FMV-Marketable Control Interest	867,094	Row 23 + Row 24				
26	Disc-Lack of Marketability	(346,837)	- Row 7 X Row 25				
27	Fair Market Value	**$520,256**	Row 25 + Row 26				
28	**Calc of Disc Rate-Regr Eq #2**						
29	Ln (FMV-Marketable Minority)	13.3728	Ln(Row 23)				
30	* X Coefficient of -.0157334352	-0.2104	Row 29 * X Coefficient-Regr #2				
31	Constant	0.4794	Constant-Regression #2				
32	Discount Rate (Rounded)	**27%**	Row 30 + Row 31				

Note: We have achieved consistency in the discount rate assumed (Row 4) and the implied discount rate (Row 32). Also the discount rates match Table II as we interpolate between $500k and $750k.

Table IV-C

Discounted Cash Flow Analysis Using Abrams' Formula-Final Iteration

Row	Description:	1996	1997	1998	1999	2000	Total
1	**Assumptions:**						
2	Base Adjusted Cash Flow	$100,000					
3	Growth Rate in Adj Cash Flow	12%	10%	9%	8%	7%	
4	Disc Rate = R [1]	29%					
5	Growth Rate To Perpetuity=G	6%					
6	Control Premium	35%					
7	Discount-Lack of Marketability	40%					
8							
9	**5 Year Forecasts**						
10							
11	Forecast Cash Flow	$112,000	$123,200	$134,288	$145,031	$155,183	
12	Present Value Factor	0.8805	0.6825	0.5291	0.4101	0.3179	
13	PV of Cash Flows	**$98,611**	**$84,086**	**$71,050**	**$59,484**	**$49,339**	**$362,569**
14							
15	**Calculation of Fair Market Value:**						
16			Formula				
17	Forecast Cash Flow 2000	$164,494	Row 11 for 2000 X (1+ Row 5)				
18	Gordon Model Cap Rate	4.9382	SQRT (1+R) / (R-G)				
19	FMV 1999-Infinity as of 1/1/2001	$812,302	Row 17 X Row 18				
20	Present Value Factor-5 Yrs	0.2799	1/(1+R)^5 [Where 5 is # yrs from 1/1/96 to 1/1/2001]				
21	PV of 2001-Infinity Cash Flow	$227,389	Row 19 X Row 20				
22	Add PV of 1996-2000 Cash Flow	362,569	Total of Row 13				
23	FMV-Marketable Minority	$589,958	Row 21 + Row 22				
24	Control Premium	206,485	Row 6 X Row 23				
25	FMV-Marketable Control Interest	796,444	Row 23 + Row 24				
26	Disc-Lack of Marketability	(318,577)	- Row 7 X Row 25				
27	Fair Market Value	**$477,866**	Row 25 + Row 26				

[1] Disc Rate = 27% (from Table IV-B, Row 32) + 2% for Company Specific Adjustments = 29%

Table V
Gordon Model Valuation
Using Newton's Iterative Process

1	**Iteration**	**Value**
2	**t**	**V(t)**
3	0	26,000
4	1	13,626,578
5	2	596,485
6	3	492,762
7	4	492,153
8	5	492,153
9		
10	**Proof of Calculation:**	
11		
12	**Discount Rate**	27.32%
13	**Gordon Multiple**	4.9215
14	**X CF = FMV**	$492,153
15		
16	**Parameters**	
17		
18	**V(0)**	**26,000**
19	**CF**	**100,000**
20	**g**	**7%**
21	**a**	**47.94%**
22	**b**	**-1.57334352%**
23		
24		
25	**Model Sensitivity**	
26	**FMV**	**Initial Guess = V(0)**
27	Explodes	74,000,000,000
28	492,153	73,000,000,000
29	492,153	26,000
30	Explodes	25,000
31		
32	Formula in Cell C8 (which is in the numbered Row 4):	
33		
34	+C7-((C7-($CF/($A+$B*@LN(C7)-$G)))/(1+($B*$CF)/(C7*($A+$B*@LN(C7)-$G)^2)))	

Note: The above formula assumes an End-Year Gordon Model. Newton's Method converges for the midyear Gordon Model, but too slowly to be of practical use.

Table VI
95% Confidence Intervals

Row	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1	**Cash Flow-CF(t)**	300,000,000		300,000,000		15,000,000		1,000,000		100,000	
2	**r (assume correct)**	11.0%		11.0%		18.00%		23.0%		27.0%	
3	**G**	9%		9%		7%		5%		5%	
4	**Cash Flow (t+1)**	327,000,000		327,000,000		16,050,000		1,050,000		105,000	
5	**Discount Rate Range**										
6	**Abrams Model**										
7	**Upper Bound [2]**	13.05%		11.54%		20.05%		25.05%		29.05%	
8	**As Calculated [1]**	11.00%		11.00%		18.00%		23.00%		27.00%	
9	**Lower Bound [2]**	8.95%		10.46%		15.95%		20.95%		24.95%	
10	**CAPM**										
11	**Upper Bound**	17.27%		17.43%		24.27%		29.27%		33.27%	
12	**As Calculated [1]**	11.00%		11.00%		18.00%		23.00%		27.00%	
13	**Lower Bound**	4.73%		4.57%		11.73%		16.73%		20.73%	
14	**Gordon Model-Abrams**										
15	**Lower Bound [3]**	26.2382		41.5159		8.3945		5.5767		4.7231	
16	**Gordon-Mid [3]**	52.6783		52.6783		9.8753		6.1614		5.1225	
17	**Upper Bound [3]**	-1994.2300	Explodes	72.1803		12.0343		6.8961		5.6037	
18	**FMV-Abrams Model**										
19	**Lower Bound [4]**	8,579,888,701	49.8%	13,575,698,719	78.8%	134,732,377	85.0%	5,855,583	90.5%	495,924	92.2%
20	**Gordon-Mid [4]**	**17,225,793,886**	100.0%	**17,225,793,886**	100.0%	**158,497,843**	100.0%	**6,469,480**	100.0%	**537,859**	100.0%
21	**Upper Bound [4]**	************************	Explodes	23,602,946,294	137.0%	193,150,821	121.9%	7,240,875	111.9%	588,385	109.4%
22	**Gordon Model-CAPM**										
23	**Lower Bound**	14.5037		14.5037		6.7581		4.8350		4.1940	
24	**Gordon-Mid**	52.6783		52.6783		9.8753		6.1614		5.1225	
25	**Upper Bound**	-29.8670	Explodes	-29.8670	Explodes	19.0817		8.6326		6.6579	
26	**FMV-CAPM**										
27	**FMV-Lower**	4,742,702,854	27.5%	4,742,702,854	27.5%	108,467,887	68.4%	5,076,704	78.5%	440,367	81.9%
28	**FMV-Mid**	**17,225,793,886**	100.0%	**17,225,793,886**	100.0%	**158,497,843**	100.0%	**6,469,480**	100.0%	**537,859**	100.0%
29	**FMV-Upper**	(9,766,501,045)	Explodes	(9,766,501,045)	Explodes	306,261,417	193.2%	9,064,199	140.1%	699,076	130.0%
30	**Verify Discount Rate**										
31	**-.0157334 * ln(FMV)**	-37.08%		-37.08%		-29.71%		-24.67%		-20.76%	
32	**Add Constant**	47.94%		47.94%		47.94%		47.94%		47.94%	
33	**Discount Rate**	**10.86%**		**10.86%**		**18.23%**		**23.27%**		**27.18%**	
34	**Rounded**	**11%**		**11%**		**18%**		**23%**		**27%**	
35	**Avg Error-Abrams**		**Explodes**		**29.1%**		**18.4%**		**10.7%**		**8.6%**
36	**Avg Error-CAPM**		**Explodes**		**Explodes**		**62.4%**		**30.8%**		**24.0%**
37											
38	**Assumptions:**										
39											
40	**Standard Errors (Regr #2):**										
41	**Abrams [5]**	**0.89%**									
42	**Abrams [5]**	**0.23%**									
43	**CAPM**	**2.72%**									

Table VII

% Valuation Error for 10% Error in Growth

	(1)	(2)	(3)
Row	**Description**	**Large Firm**	**Small Firm**
1	r	11%	27%
2	g	9%	5%
3	Gordon Model	50.0000	4.5455
4	Cash Flow	300,000,000	100,000
5	Value	15,000,000,000	454,545
6	1.1*G	9.90%	5.50%
7	Gordon Model 2	90.9091	4.6512
8	Value 2	27,272,727,273	465,116
9	V2 / V1	**1.8182**	**1.0233**
10	(V2 / V1) - 1	**81.82%**	**2.33%**
11			

12	**Sensitivity Analysis: Valuation Error for Combinations of r and g**

13							
14	**r\g**	5%	6%	7%	8%	**9%**	10%
15	11%	9.09%	13.64%	21.21%	36.36%	**81.82%**	NA
16	12%	7.69%	11.11%	16.28%	25.00%	42.86%	100.00%
17	13%	6.67%	9.38%	13.21%	19.05%	29.03%	50.00%
18	14%	5.88%	8.11%	11.11%	15.38%	21.95%	33.33%
19	15%	5.26%	7.14%	9.59%	12.90%	17.65%	25.00%
20	16%	4.76%	6.38%	8.43%	11.11%	14.75%	20.00%
21	17%	4.35%	5.77%	7.53%	9.76%	12.68%	16.67%
22	18%	4.00%	5.26%	6.80%	8.70%	11.11%	14.29%
23	19%	3.70%	4.84%	6.19%	7.84%	9.89%	12.50%
24	20%	3.45%	4.48%	5.69%	7.14%	8.91%	11.11%
25	21%	3.23%	4.17%	5.26%	6.56%	8.11%	10.00%
26	22%	3.03%	3.90%	4.90%	6.06%	7.44%	9.09%
27	23%	2.86%	3.66%	4.58%	5.63%	6.87%	8.33%
28	24%	2.70%	3.45%	4.29%	5.26%	6.38%	7.69%
29	25%	2.56%	3.26%	4.05%	4.94%	5.96%	7.14%
30	26%	2.44%	3.09%	3.83%	4.65%	5.59%	6.67%
31	**27%**	**2.33%**	2.94%	3.63%	4.40%	5.26%	6.25%