A Breakthrough In Calculating Reliable Discount Rates

I have discovered a mathematical relationship that will enable business appraisers to finally calculate discount rates for small businesses with a high degree of reliability and a minimum of research.

Fama and French

Unfortunately I cannot say that I am the first person to discover the factors involved. Professors Fama and French ("FF") found they could explain market returns well empirically with a three-factor multiple regression model using firm size, the ratio of book equity to market equity (BE/ME), and the overall market factor R_m-R_f.[1] The latter factor explained overall returns to stocks across the board, but it did not explain differences from one stock to another, or more, precisely, from one portfolio to another.

The entire variation in returns of portfolios was explained by the first two factors. FF found BE/ME to be the more significant factor in explaining the cross-sectional difference in returns, with firm size next. However, they consider both factors as proxies for risk. Furthermore, they state, "Without a theory that specifies the exact form of the state variables or common factors in returns, the choice of any particular version of the factors is somewhat arbitrary. Thus detailed stories for the slopes and average premiums associated with particular versions of the factors are suggestive, but never definitive."

Abrams' Law: The Exponential/Logarithmic Relationship of Returns and Firm Size

I discovered that return (the discount rate) strongly correlates with the natural logarithm of the value of the firm, i.e., firm size. This discovery has the following aspects and implications:

(1) The discount rate is a linear function inversely related to the natural logarithm of the value of the firm.

(2) The value of the firm is an exponential decay function, decaying with the investment rate of return (the discount rate). Alternatively, the value also decays in the same fashion with the standard deviation of returns.

These statements will become clear after explaining the underlying data.

You will please forgive my lack of modesty, but I wish to put my name to the mathematical relationship. I call it Abrams' Law.

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[1] Common Risk Factors In The Returns On Stocks And Bonds, Eugene F. Fama and Kenneth R. French, Journal of Financial Economics 33, 1993, pages 3-56. The article also includes two other factors for forecasting bond portfolios, the term premium and the default premium, but these factors are irrelevant to the purposes of this article.

The author wishes to thank Ed Murray for his considerable help in obtaining research materials and for his helpful insights.

Table I, Page 1

Table I is a two page table. The first page shows returns tabulated by decile for New York Stock Exchange (NYSE) Stocks[2] and two regression analyses of these data.

Columns 2 and 3 are the arithmetic mean returns from 1926-1993 for each decile and the standard deviation of those returns. Figure 1 shows a plot of returns as a function of risk, where we define risk as the standard deviation of returns. Note the strong relationship of the two.

Regression equation [1] from Table I shows a very strong relationship of the historical returns to their standard deviations. Restating the equation,

[1] $R = 4.98\% + .3586\, S$, where S = standard deviation of returns

The R-Square for [1] is 97.87%, the t-statistic of the slope is 19.2, the related p-value is less than 0.01%, and the standard error of the estimate is 0.47%; it's hard to do much better than that. Also, the constant of 4.98% is the regression estimate of the long-term risk-free rate. The 68 year average rate of return from 1926-1993 on long-term Treasury Bonds is 5.02%.[3] Therefore, in addition to the other robust results, our regression equation does a superb job of calculating the risk-free rate.

The problem with this result in the valuation of small businesses is coming up with a reliable standard deviation of returns. First of all, there is the problem of the wide variation that one would experience from one business to the next, especially in small businesses. The accounting anomalies alone would be difficult to effectively filter out. Each Ibbotson decile contains 158 large firms, and averages over large data sets filter out such anomalies. Additionally, we as appraisers cannot directly measure the standard deviation of returns for privately held firms. We can measure the standard deviation of income, whether adjusted or not. While that should be a major component of the standard deviation of returns, changes in interest rates would also have much impact on the returns.

Exponential Relationship of FMV and Return

Fortunately, there is a slightly less accurate, but much more practical relationship. Notice that the returns are inversely related with the market capitalization, i.e., the fair market value (FMV) of the firm.[4]

The second regression shows the return as a function of the natural logarithm of the FMV of the firm. The final regression equation is:

[2] $R = 49.27\% - [\, .0163 \times \ln(FMV)\,]$

R-Squared is 94.2%, t = -11.4, and the p-value is less than 0.01%, meaning that these results are statistically robust. The standard error of the Y Estimate being .77% means that we can be 95% sure that the regression forecast is accurate within 1.5%. Figure 2 shows both the actual and regression results for each of the 10 deciles.

Table II: Abrams Table of Equity Premia Based on FMV

For a moment, we will skip the second page of Table I and proceed to Table II, which shows the calculation of the discount rate for various values of the firm and the implied equity premium. The Ibbotson statistics are for publicly held firms, which represent marketable minority interests, which are shown in the left column.

A firm with a marketable minority FMV is $5 million would have an equity premium of 19.0%. We would add this to the current long-term bond rate of 7.5% to arrive at a discount rate of 28%, rounded. If the estimated marketable minority value for your subject company is $7 million, you can either substitute that value in [2] to come to an exact discount rate (from which you must subtract the 68-year average 5% Treasury bond rate and add the current Treasury Bond rate) or take the more practical approach and interpolate. The equity premium for a $10 million firm is 17.9%, so the correct equity premium would be somewhere between 17.9% and 19.0%. 18.5% is a reasonable guess, which when added to the 7.5% current Treasury Bond rate would yield a discount rate of 26%.

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[2] The source for all of the data is the *SBBI-1994 Yearbook*, Exhibits 54 and 57, Ibbotson & Associates, Chicago, IL.

[3] The 1994 SBBI Yearbook, page 225, shows the 1926 index for long-term Treasury Bonds as 1.000 and the 1993 index is 28.034. The average return is then $28.034^{1/68}-1 = 5.02\%$.

[4] Average capitalization is the total capitalization of each decile divided by the number of companies, or column 5 divided by column 6.

Need To Update The Table Annually

The Abrams Table of Equity Premiums should be updated annually as the Ibbotson averages change. However, 68 year averages change very slowly, so the table should remain reasonably current for a few years at a stretch.

Table I: Page 2--Comparing This Model With CAPM

Table I, Page 2 is a continuation of Page 1--the same rows, but columns immediately to the right of those in Page 1. Column 8 shows the average beta for the decile, and column 9 contains the CAPM estimate of the return. Column 10 is the error in the estimate, while column 11 is the squared error. The bottom of column 11 shows the total squared error and the standard error of the estimate, which is 2.23% for the CAPM.[5]

Column 12 is the estimated return using regression equation [2], column 13 is the error in the estimate, and column 14 is the squared error, total, and standard error. Note that the standard error of the regression equation is 0.77%, which matches regression equation [2] results on the first page of Table I. This means that unless there is systematic error in [2] (see below), we can have 68% confidence that those estimates are plus or minus .77% of the correct estimate and 95% confidence that we are within 1.5%. Note that the CAPM error is almost three times as large as [2].

Systematic Error In The Regression Equation?

The largest error in [2] is in the 10th decile, the worst place. The reason this is the worst place is that the estimates of discount rates in Table II are extrapolations of [2], not interpolations. If there is any systematic error in [2] (for which there is no strong evidence, but some possibility), then the estimated discount rates below FMVs of $39 million will be too low.

A factor mitigating the possibility of systematic underestimation of small firms is that when I repeated regression [2] using only data from 1970-1993, the regression overestimated the 10th decile.

My educated guess is that the errors in the estimates are small. Most appraisers value firms that are worth $100,000 to $1 million. A 33% discount rate for the former and a 29% rate for the latter seems quite reasonable. I have strong empirical evidence that I intend to publish in a subsequent article that supports the accuracy of these discount rates for small businesses.

An Iterative Process

Iteration seems to be a recurring theme in many of my articles, and here it comes again! If you haven't noticed yet, we have a problem of circular reasoning. We need the FMV to choose the discount rate, which we then use to discount cash flows or income to calculate ... the FMV!

Therefore it is necessary to make sure that our initial estimate of FMV is reasonably consistent with the final results. If not, then we have to keep doing it until our results are consistent. Fortunately, the discount rates remain virtually the same over large ranges of values, so this should not be much of a problem.

Tables III and III-A: Examples Of Iterations

Let's illustrate the iterative process. Tables III and III-A are identical, with the only difference being the discount rate in the assumptions of the models.

Table III is a very simple Discounted Cash Flow (DCF) Analysis of our hypothetical firm. Let's suppose that we make a terrible guess and say that this firm is worth around $3 million (of course it is, just ask the owner!). Looking up the equity premium in Abrams' Table in Table II, we would add the premium of 19.9% + 7.5% (Current Long Term Government Bond Rate) = 27% (rounded), which is the discount rate that we have used in Row 4 of Table III. Our model generates a marketable minority value of $642,292 (Row 23), slightly short of our initial estimate.

Table III-A Second Iteration:

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[5] The standard error is the square root of the sum of the squared errors, divided by the square root of 8, the latter of which is number of observations - # regressors - 1, or 10-1-1=8.

Looking up a $642,292 fair market value in Table II, the correct equity premium is approximately 22.5% (by interpolation), which when added to 7.5% comes to 30%. We use the 30% discount rate in Table III-A, which generates a marketable minority value of $567,063. This value and the discount rate which generated it are sufficiently consistent, as we can see in Table II, so we can stop. Of course, we must then calculate the appropriate control premium and discount for lack of marketability to come to the FMV of a privately-held firm, which appears in Rows 24-27. The 40% discount for lack of marketability is based on an article which I am publishing in Business Valuation Review, currently scheduled to print in the September, 1994 issue.

Consistency and Levels of Value: It is important to remain consistent in which level of fair market value we are comparing to the table. Since the returns in the table were generated from the NYSE, which are marketable minority returns, we must compare the DCF's fair market value at that same level, even though it is not the ultimate fair market value that will go into our opinion letter. Also, I had a strong prejudice that was hard to shake, which was that a DCF always generates an illiquid control interest value. This methodology does not do that, because the data from which it is derived is a marketable minority.

Fama, French, and the Traditional CAPM

The FAMA and French results show that beta correlates with both firm size and the book equity/market equity. For example, the beta for the portfolio of stocks for the smallest firms and the smallest book-to-market equity ratio quintiles was 1.4 in the traditional one-factor CAPM regression, while the beta for the largest firm quintile and largest book-to-market equity quintile was .89. In the three-factor model, while the respective betas were 1.04 and 1.06. The three factor model renders beta powerless to explain the difference in returns between one firm and another. It only explains an equity premium common to all stocks. My model even eliminates that use of beta.

Implications For Traditional Valuation Techniques

The strong size effects found by FF as well as myself in this article strongly suggest that the traditional one-factor CAPM model is obsolete. As FF say on page 54, "Many continue to use the one-factor Sharpe-Lintner model to evaluate portfolio performance and to estimate the cost of capital, despite the lack of evidence that it is relevant. At a minimum, the results here and in Fama and French, The Cross-Section of Expected Stock Returns, Journal of Finance 47, 427-465 should help to break this common habit."

Think about the usual way to calculate the CAPM. We average the betas of many different firms in the industry, which vary considerably in size. Ignoring the size effect means that the average beta implicitly averages across the sizes in the industry (and not very well at that, because this article shows it is the logarithm of size, not the absolute size, that is the best factor to use in calculating discount rates) and then apply that to a firm that is probably less than .01% of the industry average, also without correction for size, and hence, risk.

This flaw also applies to using a Market Comparison Approach. We average Price Earnings multiples (and/or Price Cash Flow multiples, etc.) for the various firms in the industry without correcting for size, which gives us implicitly a flawed PE multiple for the average size "comparable" firm and apply that to a tiny private firm.

This process is erroneous. I believe the size effect as captured in this article will enable business appraisers to dispense with CAPM and use firm size as the base discount rate before adjustments for qualitative factors. With regard to developing PE Multiples, it is imperative to include firm size and forecast growth as factors in a regression. Simple or weighted average PE multiples are erroneous. However, cross-sectional regressions at a single point in time will be subject to considerably more error in the size effect than a 68 year time series, as at any point in time the size effect is likely to diverge from the long-run average logarithmic relationship.

Lynch, Jones, and Ryan issued a study where they calculated the P/E ratio from a two-factor regression, with Long Term Growth Rate and Market Capitalization as the independent variables. Their r-squares were 89% for December 1989 data and 73% for November 1990 data. Using the natural logarithm of market capitalization instead, I found the same data yields r-squares of 91% for each data set, a marginal increase in explanatory power for the first regression and a significant increase in explanatory power for the second regression.

What would r-squares be with our traditional Price Earnings Multiple calculations? Probably not more than 30% to 50%.

The Exponential/Logarithmic Relationship Is Primarily Long-Run

An important question is why did two of the greatest researchers in this generation not find the logarithmic relationship found by an MBA? The reason is that FF were working with 1963-1990 data, and the logarithmic relationship works beautifully over the 68 years covered by Ibbotson & Associates (1926-1993), but the relationship wavers over shorter time periods.[6]

I repeated regressions [1] and [2] for 1970-1993 and found that the absolute value of the firm was a better regressor than its log. Additionally I added a third regression, [2a], which is the same as [2], except that the independent variable is the FMV, not its logarithm. R-squares for different regressions are:

R-Squares

	Regression [1]	Regression [2]	Regression [2a]
	R= a + b (Std Dev)	R= a + b ln(FMV)	R= a + B FMV
1970-1993	40.3%	50.4%	82.5%
1970-1979, 1990-1993	79.8%	79.0%	79.0%

Note the poor results of Regression [1] for 1970-1993. Looking only at this time period, we would say that it doesn't appear that historical risk, as measured by the standard deviation of returns, is a very good explanatory variable for returns. Similarly, the log of the FMV of the firm has mediocre ability to explain returns, while the actual fair market value of the firm has much better explanatory power, at 82.5% R-square. FF mention that the 1980's had rather anomalous results, so I repeated the regressions for the same data set, minus the 1980's. Now all three regressions seem to work equally as well, though less well than the 68 year averages.

My understanding of this is that the long-run relationship is akin to viewing an impressionist painting. If you stand too closely, all you see is a series of streaks and splotches. It takes distance to see the patterns. Similarly, the exponential/logarithmic relationship I found manifests over long periods of time and may not work well over certain shorter periods of time.

The question for business appraisers is whether or not the long-run information is the best to use. The answer is a qualified yes.

First, Shannon Pratt shows how owners of privately-held firms are more concerned with the long run than are investors in the public market: "I think it is fair to assume that the typical buyer of a closely held company or an interest in one approaches the investment with a longer time horizon in mind than does the typical buyer of publicly traded stocks. This implies that investors in closely held companies are likely to be less than fully responsive to short-term swings in the public stock market's expected rate of return. In other words, they may be more likely to base their expected rate of return on the public market's average or normalized rate of return over some time period than on its expected rate of return as of any specific date or short-term period. In fact market evidence bears this out..."[7]

Ibbotson shows that there is very little autocorrelation of the data[8], which suggests that the data is independently distributed over time. This would suggest that the longer the data period, the better, and the full 68 years of the NYSE is better than any shorter period. Unfortunately it is not quite that simple. There is some significant autocorrelation in the smallest deciles, which suggests that it may be appropriate to place some greater weight on more recent data. Nevertheless, the autocorrelation is not yet well understood in the smaller NYSE firms, and it is premature to draw conclusions as to the implications. I would speculate that the differences for appraisers would be negligible, but an investor with a lot of money in the market may want to treat the logarithmic relationship as a Bayesian prior and revise the forecast with more recent data.

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[6]To balance out my lack of humility in calling this relationship Abrams Law, I feel it important to point out that I was very lucky, having practically tripped over the answer. Fama and French are far more sophisticated than I am, and there is no way I could duplicate their methodology. They have easy access to the CRSP data from the University of Chicago, while I am practically limited to Ibbotson's summarization of that data. Additionally, I do not have the sophisticated training that they have. My limitations in training, funding, and access to all of the detail worked in my favor.

[7]Valuing A Business, 2nd Ed., Business One Irwin, Homewood, IL, page 51.

[8]Ibid., page 126.

What Does The Exponential Relationship Mean?

Let's try to get an intuitive feel for what an exponential relationship means and why that might make intuitive sense. I develop the mathematics in the Appendix. Equation [A5] shows that the fair market value of the firm is an exponentially declining function of risk, as measured by the standard deviation of returns. Repeating [A6], $FMV = Ae^{kS}$, $k < 0$. Because we find that risk itself is primarily related to the size of the firm, we come to a similar equation for size. Repeating [A10], we see that $FMV = Ce^{mr}$, $m < 0$.

In physics, radioactive minerals such as uranium decay exponentially. That means that a constant proportion of uranium decays at every moment. As the remaining portion of uranium is constantly less due over time due to the radioactive decay, the amount of decay at any moment in time or during any finite time period is always less than the previous period. A graph of the amount a uranium remaining over time would be a downward sloping curve, steep at first, and increasing shallow over time. Figure 3 shows an exponential decay curve.

It appears the same is true of the value of firms. Instead of decaying over time, their value decays over risk. Because it turns out that risk is so closely related to size and the rate of return is so closely related to size, the value also decays exponentially with the market rate of return, i.e., the discount rate. The graph of exponential decay in value over risk has the same general shape as the uranium decay curve.

Imagine the largest ship in the world sailing on a moderately stormy ocean. You as a passenger might hardly feel the effects of the storm. If instead you sailed on a slightly smaller ship, you would feel the storm a bit more. As we keep switching to increasingly smaller ships, the storm feels increasing powerful. The smallest ship on the NYSE might be akin to a 35 foot cabin cruiser, while we appraisers have to value little paddle boats, the passengers of which would be in danger of their lives while the passengers of the General Electric boat would hardly feel the turbulence.

That is my understanding of the principal underlying the size effect. Size offers diversification of product and service. Size reduces transactions costs in proportion to the entity, e.g., the proceeds of floating a $1 million stock issue after flotation costs are far less in percentage terms than floating a $100 million stock issue. Large firms have greater depth and breadth of management, and they have greater staying power. Even the chances of beating a bankruptcy exist for the largest businesses. Remember Chrysler? If it were not a very big business, the government would never have jumped in to rescue it. The same is true of the S&Ls. For these and other reasons, the returns of big businesses fluctuate less than small businesses, which means that the smaller the business, the greater the risk, the greater the return.

The FMV of a firm or portfolio declining exponentially with the discount rate/risk is reminiscent of a continuous time present value formula, where the Present Value = Principal * e^{-rt}; in this case, though, instead of travelling through time we are travelling though expected rates of return/risk.

Figure 3 is also true when we relabel the vertical axis as return (discount rate) and the horizontal axis as fair market value, as we can see in regression [2]. When we subtract a logarithmic curve (with its characteristic shape) from a constant, the resulting curve resembles Figure 3.

Adjustments To Abrams' Table of Equity Premia

Is Table II the last word in calculating discount rates? No, but it is the best starting place. Table II is an extrapolation of NYSE data to privately held firms. While the results appear very reasonable to me, I would like to see a similar regression for NASD data. Unfortunately I don't have access to the data.

Privately held firms are generally owned by people who are not well diversified. Table II was derived from portfolios of stocks that were diversified in every sense except for size, as size itself was the method of sorting the deciles. The owner of the local bar is probably not well diversified, nor is his probable buyer. The appraiser may want to add 2% to 5% to the discount rate implied by Table II to account for that. On the other hand, a $1 million FMV firm is likely to be bought by a well diversified buyer and may not merit increasing the discount rate.

Another common adjustment I can imagine to Table II discount rates would be for depth and breadth of management compared to other firms of the same size. In general, Table II already accounts for the size effect. Nobody expects a $100,000 FMV firm to have three Harvard MBAs running it, but there is still a difference between a complete one-man show and a firm with two talented people. In general, this methodology of calculating discount rates will increase the importance of comparing the subject company to its peers via RMA Associates or similar data.

Differences in leverage between the subject company and its RMA peers could well be another common adjustment.

Discounted Cash Flow or Net Income?

It seems clear to me that since the market returns are based on the cash dividends and the market price at which one can sell his stock, the discount rates in this article properly should be applied to cash flow, not to income. However, we appraisers are often working with quick and dirty valuations, and we often don't want to bother estimating cash flow. I have seen suggestions in Business Valuation Review that we can increase the discount rate and apply this to net income. I would prefer to make a quick and dirty estimate of how much lower cash flow will be than income and apply this to cash flow, for the reason that now we have an accurate estimate of the discount rate and therefore it should be left at its correct level. Forecasting both income and cash flow is educated guesswork, where we have much less knowledge than that of discount rates, so I think it best to shoot from the hip on the cash flow forecast. Nevertheless, either approach will work. I would like to see some empirical research on the differences of net income and cash flow.

More Research Is Needed

I expect that this article will spawn many research projects, as this is certainly not the last word on the topic. There is much work to be done. I would like to see more categories in Ibbotson, i.e., instead of deciles with 158 companies each, perhaps the ASA can convince Ibbotson & Associates to give us 20 to 30 categories with 50 to 75 companies in each. The finer breakdown may help us determine whether there is any systematic underestimation of the discount rate in equation [2]. It should also refine the estimate of [2] and improve its r-squared.

I would like to repeat [2] with sales as a measure of size instead of FMV. If sales was as good a proxy for risk as value is, then it would eliminate the necessity for iterations in the valuation. My suspicion is that FMV contains some important information that would be missing in sales. The question is how much, and the answer must be determined empirically.

Another areas for further research would be to look for industry effects and to investigate the differences between behavior of each decile as a portfolio and the individual stocks comprising the portfolio. In other words, to what extent is firm-specific and industry-specific risk (if any exists) diversified away in the portfolio? How much firm-specific risk is there with our new understanding of size as a proxy for risk?

I would like to repeat [2] using leverage as a regressor to see its effects on return and value (any sponsors?).

Use In Real Estate [9]

I am not a real estate appraiser, so read these words with an appropriate grain of salt. It would seem to me that we can use an arbitrage argument to assert that the rates of return in this article are applicable to real estate. If we consider two investments side-by-side, a business and a shopping center, it would seem to me that the returns (discount rate) must be the same if risk or value is the same. Otherwise, one or the other would constitute an inefficient investment. People would shift their investments either from the stock market to real estate or vice-versa in order to invest efficiently, and the market would then force prices and returns of stocks and real estate back to an efficient frontier.

The 94% r-square for Regression #2 tells us that there is little room for leverage or any other factor to explain returns, unless there are systematic differences in leverage between the 10 deciles, which we certainly cannot assume.[10]

Since the rates of return in Table II are marketable minority interests, one must make adjustments in the real estate value to compare the ex-post answer and the ex-ante assumption of value on a consistent basis, which will demand some sort of conversion of the real estate level of value to a marketable minority equivalent, whatever that might mean in real estate appraisal.

Conclusion

The robust statistical evidence supporting Abrams' Law enables our profession to do something it has never been able to do before--to say that we know[11] what the discount rate is for small firms. Even if many of us had hunches of the range, there was never any strong evidence. Now there is.

Not only is Abrams' Table of Equity Premiums three times more accurate than CAPM for NYSE firms, it is far more accurate than CAPM for smaller firms, as CAPM gives us no clue as to the magnitude of sub-NYSE small firm premia. Also, the Abrams Table of Equity Premia requires none of the market research that CAPM requires. Not only is it better, it's cheaper. Now you can send your checks to ...

[9] This section has been added to the original article for the benefit of the National Association of Real Estate Appraisers, by whom the author was solicited to reprint the article. Additionally, Tables II, III, III-A, and the related commentary in this article contain minor corrections to the original article.

[10] The data was not easily available for analysis.

[11] Within a very narrow range

Mathematical Appendix

Definitions:

r = return of a portfolio
S = standard deviation of returns of the portfolio

a_1, a_2, b_1, and b_2 are parameters determined in regression equations [1] and [2].

a_1 is an estimate of the risk-free rate. With a_1 = 4.98% and the 67-year average risk-free rate = 5.02%, a_1 is a very accurate estimate. b_1 is the price of risk, the cost in required return of each additional increment of standard deviation in returns. a_2 is the return (discount rate) for a firm with a value of $1, as a_2 is the y-intercept for an x value of ln(1) = 0. Finally, b_2 is the price of risk as measured by the log of the value of the firm rather than by the standard deviation of returns.

We see from [1] that the return on a portfolio of securities (each decile is a portfolio) varies positively with the risk of the portfolio, or:

[A1] $$r = a_1 + b_1 S$$

This equation is not directly observable for privately held firms. Therefore we use the next equation as a proxy for risk.

[A2] $$r = a_2 + b_2 \ln(FMV), \quad b_2 < 0$$

Equating the right-hand sides of both equations and solving, we see how we are implicitly using the size of the firm as a proxy for risk.

[A3] $$S = [(a_2 - a_1)/b_1] + (b_2/b_1)\ln(FMV)$$

$(a_2 - a_1)$ is the equity premium for a $1 firm, i.e., the minimum value firm. When divided by b_1, the price of risk for each increment of standard deviation, the term in brackets is the standard deviation of a $1 firm. We then reduce our estimate of the standard deviation by the ratio of the relative prices of risk in size divided by the price of risk in standard deviation, and multiply that ratio by the log of the size of the firm. In other words, we start with the maximum risk, a $1 firm, and reduce the standard deviation by the appropriate price times the log of the value of the firm.

Rearranging [A3], we get

[A4] $$\ln(FMV) = \frac{(a_1 - a_2) + b_1 S}{b_2}$$

Raising both sides of the equation as powers of e, the natural exponent, we get:

[A5] $$FMV = e^{[(a1-a2)+b1\,S]/b2}\text{, or}$$

[A6] $$FMV = Ae^{kS}\text{, where } A = e^{(a1-a2)/b2},\ k = b_1/b_2 < 0$$

Here we see that the value of the firm or portfolio declines exponentially with risk, i.e., the standard deviation.

Unfortunately, the standard deviation of most private firms is unobservable, since there are no reliable market prices. Therefore, we must solve for the value of a private firm another way. Restating [A2],

[A7] $$r = a_2 + b_2 \ln(FMV)$$

[A8] $$\ln(FMV) = \frac{(r - a_2)}{b_2}$$

Raising both sides by e, we get

[A9] FMV = $e^{(r-a2)/b2}$, or

[A10] FMV = Ce^{mr} , where $C = e^{-a2/b2}$ and $m = 1/b_2$

This shows the FMV of a firm or portfolio declines exponentially with the discount rate. This is reminiscent of a continuous time present value formula; in this case, though, instead of travelling through time we are travelling though expected rates of return. The same is true of [A6], where we are travelling through risk.

Table I

NYSE Data By Decile & Statistical Analysis

(2)	(3)	(4)	(5)	(6) =(5)/158 Firms	(7)
Y	X1	Recent Mkt			X2
Arith		Capitali-	Decile	Firm	
Mean	Std Dev	zation	Capitalization	Capitalization	Ln(FMV)
11.15%	**18.89%**	61.98%	$2,733,565,920,000	$17,301,050,127	**23.5740**
13.28%	**22.66%**	16.11%	710,515,440,000	4,496,933,165	**22.2267**
14.09%	**24.60%**	8.45%	372,678,800,000	2,358,726,582	**21.5814**
14.70%	**27.10%**	5.06%	223,166,240,000	1,412,444,557	**21.0686**
15.72%	**27.94%**	3.43%	151,276,720,000	957,447,595	**20.6798**
15.71%	**29.08%**	2.13%	93,941,520,000	594,566,582	**20.2033**
16.15%	**31.62%**	1.42%	62,627,680,000	396,377,722	**19.7979**
17.09%	**35.34%**	0.87%	38,370,480,000	242,851,139	**19.3080**
18.15%	**38.05%**	0.42%	18,523,680,000	117,238,481	**18.5797**
22.29%	**47.48%**	0.14%	6,174,560,000	39,079,494	**17.4811**
3.01%					
12.03%	**20.34%**	**100.01%**	**$4,410,400,000,000**		

Regression [1]: Return = F(Std Dev. of Returns)

Constant	4.98%
Std Err of Y Est	0.47%
R Squared	97.87%
No. of Observations	10
Degrees of Freedom	8
X Coefficient	35.86%
Std Err of Coef.	1.87%
T	19.2
P	<.01%

Regression [2]: Return = F[LN(Mkt Capitalization)]

Constant	49.27%
Std Err of Y Est	0.77%
R Squared	94.20%
No. of Observations	10
Degrees of Freedom	8
X Coefficient	-1.63%
Std Err of Coef.	0.14%
T	-11.4
P	< .01%

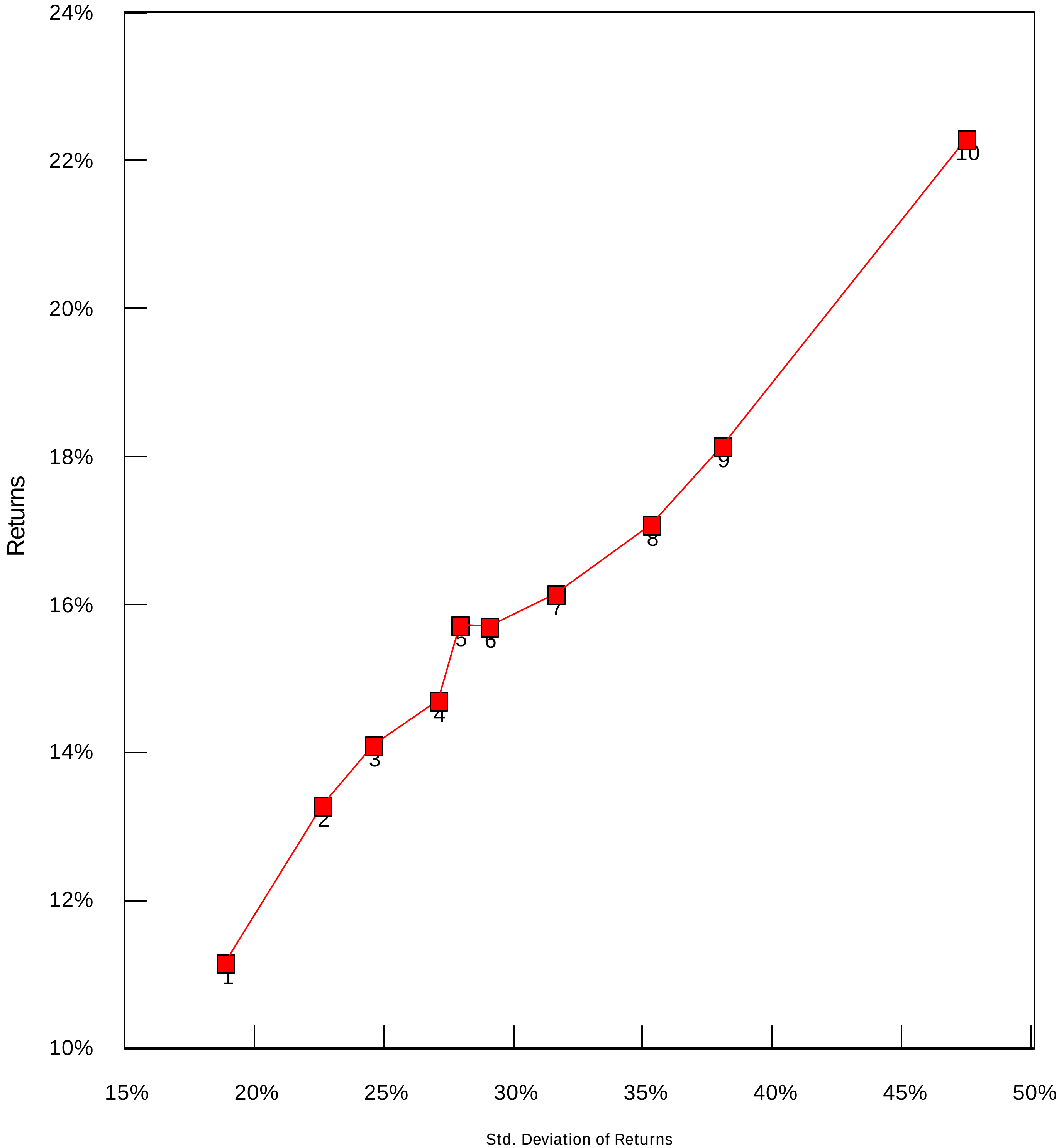
Table I (Cont'd)

Comparing Regression [2] To CAPM

(8)	(9)	(10)	(11)	(12)	(13)	(14)
				Regr [2]	Regr [2]	
Beta	CAPM E(R)	CAPM Error	Sq Error	Estimate	Error	Sq Error
0.90	11.48%	-0.33%	0.0011%	10.73%	-0.42%	0.0018%
1.04	12.49%	0.79%	0.0063%	12.93%	-0.35%	0.0012%
1.10	12.92%	1.17%	0.0137%	13.98%	-0.11%	0.0001%
1.14	13.21%	1.49%	0.0223%	14.82%	0.12%	0.0001%
1.17	13.42%	2.30%	0.0527%	15.46%	-0.26%	0.0007%
1.19	13.57%	2.14%	0.0459%	16.24%	0.53%	0.0028%
1.25	14.00%	2.15%	0.0462%	16.90%	0.75%	0.0056%
1.29	14.29%	2.80%	0.0785%	17.70%	0.61%	0.0037%
1.36	14.79%	3.36%	0.1128%	18.89%	0.74%	0.0055%
1.47	20.88%	1.41%	0.0198%	20.69%	-1.60%	0.0257%
		Totals ->	**0.3992%**			**0.0473%**
		Std Error ->	**2.23%**			**0.77%**

Note: 1926-1993 Average Long-Horizon Equity Premium is 7.2%; 1926-1993 Avg Long-Term Gov't Bond Rate = 5.0%. CAPM forecast is R = 5% + (Beta X 7.2%). Add 5.3% small stock returns for 10th decile. Above returns are arithmetic means.

Fig. 1
Return Vs. Risk



Note Box #'s are Decile #'s

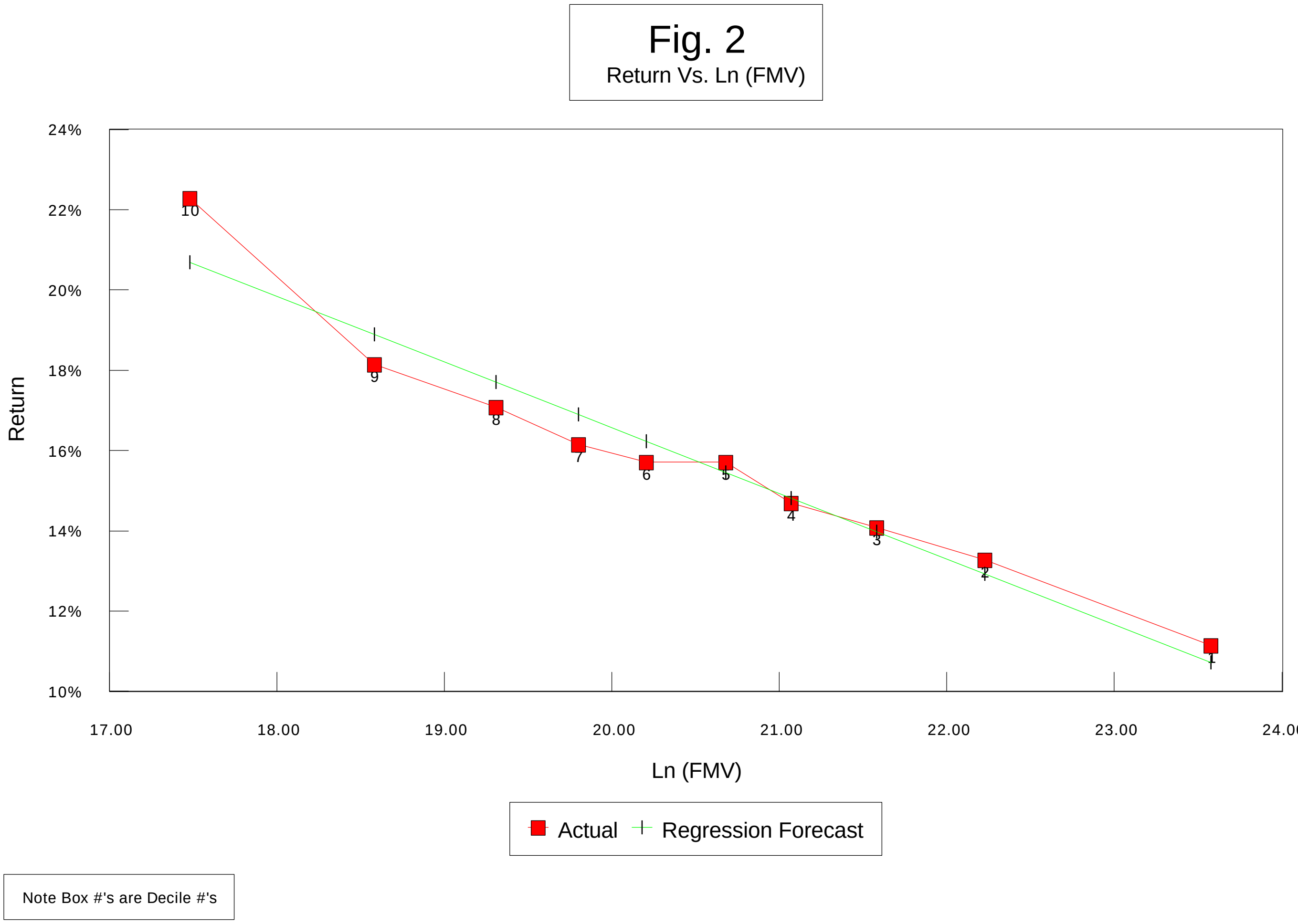


Fig. 2
Return Vs. Ln (FMV)
24%
22%
20%
18%
16%
14%
12%
10%
Return
10
9
8
7
6
5
4
3
2
1
17.00
18.00
19.00
20.00
21.00
22.00
23.00
24.00
Ln (FMV)
Actual
Regression Forecast
Note Box #'s are Decile #'s

Fig. 3
Exponential Decay Curve

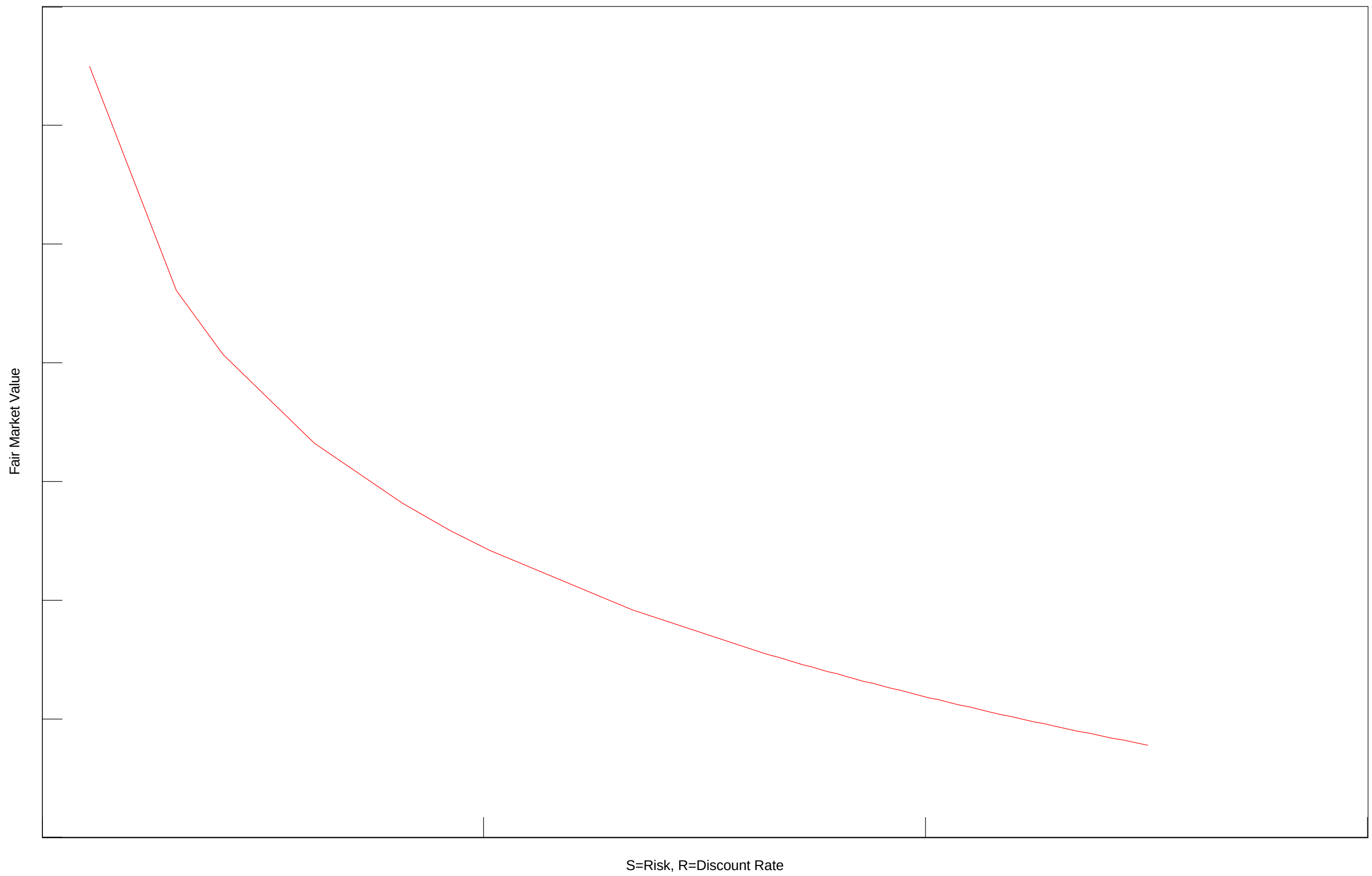


Table II

Abrams Table of Equity Premia Based on FMV

Regression Results		Implied
Mktable Min FMV	Implied R	Eqty Prem
$10,000,000,000	11.6%	6.6%
$1,000,000,000	15.4%	10.4%
$100,000,000	19.2%	14.1%
$50,000,000	20.3%	15.3%
$10,000,000	22.9%	17.9%
$5,000,000	24.0%	19.0%
$3,000,000	24.9%	19.9%
$1,000,000	26.7%	21.7%
$500,000	27.8%	22.8%
$300,000	28.6%	23.6%
$100,000	30.4%	25.4%
$50,000	31.6%	26.6%
$30,000	32.4%	27.4%
$10,000	34.2%	29.2%
$5,000	35.3%	30.3%
$1,000	38.0%	33.0%
$1	49.3%	44.2%

Assumptions:

Majority Int Premium	35%
Long-Term Gov't Bond Rate [Note]	5.02%

Note: L-T Bond Rate from Ibbotson, Page 225. 1993 return index is 28.034 and 1926 is 1.000. Avg return is 28.034^(1/68)-1 = 5.02%.

Table III

Discounted Cash Flow Analysis Using Abrams' Table Of Equity Premia-1st Iteration

Row	Description:	1994	1995	1996	1997	1998	Total
1	**Assumptions:**						
2	Base Adjusted Cash Flow	$100,000					
3	Growth Rate in Adj Cash Flow	12%	10%	9%	8%	7%	
4	Discount Rate = R	27%					
5	Growth Rate To Perpetuity=G	6%					
6	Control Premium	35%					
7	Discount-Lack of Marketability	40%					
8							
9	**5 Year Forecasts**						
10							
11	Forecast Cash Flow	$112,000	$123,200	$134,288	$145,031	$155,183	
12	Present Value Factor	0.8874	0.6987	0.5502	0.4332	0.3411	
13	PV of Adj Net Inc After Taxes	**$99,384**	**$86,081**	**$73,880**	**$62,827**	**$52,933**	**$375,105**
14							
15	**Calculation of Fair Market Value:**						
16			**Formula**				
17	Forecast Cash Flow 1999	$164,494	Row 11 for 1998 X (1+ Row 5)				
18	Gordon Model Cap Rate	5.3664	SQRT (1+R) / (R-G)				
19	FMV 1999-Infinity as of 1/1/99	$882,741	Row 17 X Row 18				
20	Present Value Factor-5 Yrs	0.3027	1/(1+R)^5 [Where 5 is # yrs from 1/1/94 to 1/1/99]				
21	PV of 1999-Infinity Cash Flow	$267,187	Row 19 X Row 20				
22	Add PV of 1994-1998 Cash Flow	375,105	Total of Row 13				
23	FMV-Marketable Minority	$642,292	Row 21 + Row 22				
24	Control Premium	224,802	Row 6 X Row 23				
25	FMV-Marketable Control Interest	867,094	Row 23 + Row 24				
26	Disc-Lack of Marketability	(346,837)	- Row 7 X Row 25				
27	Fair Market Value	**$520,256**	Row 25 + Row 26				

Table III-A

Discounted Cash Flow Analysis Using Abrams' Table Of Equity Premia-2nd Iteration

Row	Description:	1994	1995	1996	1997	1998	Total
1	**Assumptions:**						
2	Base Adjusted Cash Flow	$100,000					
3	Growth Rate in Adj Cash Flow	12%	10%	9%	8%	7%	
4	Discount Rate = R	30%					
5	Growth Rate To Perpetuity=G	6%					
6	Control Premium	35%					
7	Discount-Lack of Marketability	40%					
8							
9	**5 Year Forecasts**						
10							
11	Forecast Cash Flow	$112,000	$123,200	$134,288	$145,031	$155,183	
12	Present Value Factor	0.8771	0.6747	0.5190	0.3992	0.3071	
13	PV of Adj Net Inc After Taxes	**$98,230**	**$83,118**	**$69,691**	**$57,897**	**$47,654**	**$356,591**
14							
15	**Calculation of Fair Market Value:**						
16			**Formula**				
17	Forecast Cash Flow 1999	$164,494	Row 11 for 1998 X (1+ Row 5)				
18	Gordon Model Cap Rate	4.7507	SQRT (1+R) / (R-G)				
19	FMV 1999-Infinity as of 1/1/99	$781,468	Row 17 X Row 18				
20	Present Value Factor-5 Yrs	0.2693	1/(1+R)^5 [Where 5 is # yrs from 1/1/94 to 1/1/99]				
21	PV of 1999-Infinity Cash Flow	$210,472	Row 19 X Row 20				
22	Add PV of 1994-1998 Cash Flow	356,591	Total of Row 13				
23	FMV-Marketable Minority	$567,063	Row 21 + Row 22				
24	Control Premium	198,472	Row 6 X Row 23				
25	FMV-Marketable Control Interest	765,536	Row 23 + Row 24				
26	Disc-Lack of Marketability	(306,214)	- Row 7 X Row 25				
27	Fair Market Value	**$459,321**	Row 25 + Row 26				